Policy with Stochastic Hysteresis*

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Abstract

This paper studies stochastic hysteresis – general dependence on the path of past decisions and shocks. We develop a new methodology for deriving the explicit dynamics of optimal policy with path-dependence and show that stochastic hysteresis changes optimal policy both qualitatively and quantitatively. We showcase our methodology by deriving new results for optimal policy with stochastic habits, tipping points, robustness concerns, limited commitment, and dynamic private information.

Keywords: stochastic hysteresis, path-dependence, optimal policy, habits, tipping points, robustness, limited commitment, dynamic private information.

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1 Introduction

We develop a new methodology to analyze environments with general dependence on past shocks and choices which we refer to as stochastic hysteresis. Our benchmark environment is a habit economy which is a broad representative example of settings with path-dependence. We explicitly characterize the dynamics of optimal policy and show how stochastic hysteresis changes policy prescriptions both qualitatively and quantitatively. We further use these methods to tractably analyze robust policy design, and to derive new insights in models with limited commitment and dynamic private information, in the presence of stochastic hysteresis.

We start by analyzing a representative agent production economy with stochastic productivity, external habit formation, and labor supply similar to Ljungqvist and Uhlig (2000) in continuous time. In their classic work, Stigler and Becker (1977) propose that preferences depend on the trajectory of past consumption. Our formulation allows habits to have stochastic hysteresis, that is, to directly depend both on past choices and the trajectory of uncertain shocks. The shocks can be thought of as experiences and circumstances that directly influence habits beyond merely the quantity consumed. While we present our methodology for an economy with habit formation, habits can be understood generally as any dependence of the economy on past choices and shocks.[1]

Our primary objective is to characterize the dynamics of optimal policy. Optimal policy in an environment with external habit formation corrects the externality that arises because agents do not internalize the effect of their decisions on the habits, and therefore welfare, of others. Specifically, an increase in current consumption affects habits along all future stochastic trajectories. The main challenge in characterizing optimal policy is that the conditional expectation of the future marginal effects on habits, and hence optimal policy, is path-dependent. In general, path-dependence does not allow to describe optimal policy dynamics using the standard Ito formula as it applies only to functions of the state and not to functions of a path.

The main new tool that we develop – the total derivative formula for conditional expectation processes – allows us to characterize the dynamics of general path-dependent processes. This formula is broadly applicable to a variety of settings where conditional expectations processes are

[1] In Section 2 we show stochastic hysteresis arises in several important classes of economies: in macro-climate models (Nordhaus, 1992, 1993; Golosov, Hassler, Krusell, and Tsyvinski, 2014) as path-dependency in climate variables, in firm investment models (Arrow, 1964; Rogerson, 2008) as path-dependent stochastic depreciation, and in knowledge accumulation models (Becker, 1962; Ben-Porath, 1967; Heckman, 1976; Arrow, 1962; Uzawa, 1965; Lucas, 1988) as path-dependent returns on education. Stochastic habits appear in, for example, Chetty and Szeidl (2016) where the habit weights on aggregate consumption are stochastic and depend on the distribution of agents.

central. The total derivative formula provides a drift and diffusion decomposition of the dynamics of optimal policy. The first term in the total derivative formula gives the drift of the process which represents how the conditional expectation evolves with respect to time. The second term gives the diffusion coefficient and represents the sensitivity of the conditional expectation with respect to changes in the underlying stochastic process. To derive this sensitivity we apply the concepts of Malliavin calculus, a mathematical toolkit of the stochastic calculus of variations. We establish that the sensitivity of the conditional expectation is the Malliavin derivative of the expected marginal habits. Given an initial condition, the drift and diffusion decomposition provides a full description of optimal policy along any path. The drift and diffusion decomposition thus breaks down complex interactions of time and uncertainty in the conditional expectation into a drift and a diffusion term.

To show the implications of stochastic hysteresis on policy design, we develop a number of examples. Our first example focuses on stochastic habits that reduce the significance of past consumption based on the level of dissimilarity between current productivity and past experiences. Habits formed under vastly different circumstances are less relevant. We refer to this notion of experiencing something different from what one is used to as unfamiliarity. We show that optimal policy can be viewed as a price of an asset that pays a lower dividend in unfamiliar states. Optimal policy includes a path-dependent component determined by unfamiliarity, that is, by the extent of dissimilarity between future states and past experiences. The total derivative formula provides the value of the assets, and hence optimal policy, in closed form.[2]

Using the analytical characterization of optimal policy we quantitatively evaluate optimal policy with stochastic hysteresis under unfamiliarity and compare our findings to those of Ljungqvist and Uhlig (2000). Our first quantitative result is that optimal policy is neither procyclical nor countercyclical. The reason is that unfamiliarity, measured by the distance between the current productivity shock and past experiences, is symmetric. Good and bad times are equally unfamiliar which implies that the habit externality and, hence, optimal policy are identical in these times. The second quantitative result is stochastic hysteresis in optimal policy, that is, lasting

[2] Our formulation of unfamiliarity is consistent with Gilboa and Schmeidler (1995) and Billot, Gilboa, Samet, and Schmeidler (2005) which provide a foundation for incorporating similarity of previous experiences in agents' decision-making. Recent work on memory (see Malmendier and Wachter (2022) for a review) emphasizes the importance of similarity with prior experiences (Mullainathan, 2002; Bordalo, Gennaioli, and Shleifer, 2020; Wachter and Kahana, 2021), and the long-lasting influence of personal experiences (see, for example, Malmendier and Nagel (2011) for the effects of stock market experiences and Malmendier and Nagel (2016) for the effects of inflation experiences).

dependence on past shocks. Even when current and expected productivities are identical over a large part of the sample, optimal policy qualitatively differs. The total derivative formula allows us to derive the main factors behind these differences. Specifically, we show that the behavior of the diffusion coefficient governed by the Malliavin derivative of unfamiliarity plays a central role in determining the qualitative differences. These two results stand in contrast to Ljungqvist and Uhlig (2000) where the habit externality is cyclical and depends only on the current state. Finally, we show that unfamiliarity has significant welfare consequences. Since anticipated productivity shocks make current experiences less relevant for the future, an increase in current consumption imposes a smaller externality compared to the specification without unfamiliarity.

Our second main example analyzes tipping points, which are endogenous thresholds that cause a significant shift in the behavior of an economy or agents. Specifically, we model a tipping point by allowing habits to change once a new maximum in the trajectory of shocks is attained. We provide a closed-form drift and diffusion decomposition that gives the dynamics of optimal policy with tipping points. The diffusion coefficient is of particular interest as we explicitly show how a change in the trajectory of the stochastic process affects the probability of crossing a tipping point.

We then showcase how our methodology can be applied to three important environments: robustness, limited commitment, and dynamic private information. First, we show how to tractably analyze optimal policy with robustness concerns. Robustness may be of particularly concern in our path-dependent framework since habits directly depend on the shock trajectory and are thus directly affected by misspecification. We consider two approaches to robust policy design. The first approach studies policy design with concerns for robustness in the sense of Hansen and Sargent (2001, 2008). We show that robust optimal policy is represented by exponentially tilted habit coefficients that account for the robustness considerations. Using the total derivative formula we establish that the drift of optimal policy is adjusted by the level of the tilt, while the diffusion coefficient is adjusted by both the level of the tilt and its sensitivity to uncertainty as measured by the Malliavin derivative. The second approach characterizes the sensitivity of optimal policy to local misspecification, drawing connections to robust statistics (Hampel, 1974; Huber, 1981). We show that optimal policy has to be corrected by the effects of misspecification as measured by its Malliavin sensitivity. Both the characterization of optimal robust policy and its dynamics in a general path-dependent environment as well as the use of Malliavin derivatives to determine the first-order effects of misspecification are new to the robustness literature.

Second, we incorporate limited commitment frictions in the economy with stochastic habits. Limited commitment entails additional path-dependence since an increase in current consumption relaxes past participation constraints (Thomas and Worrall, 1988; Kehoe and Levine, 1993; Kocherlakota, 1996; Marcet and Marimon, 2019). The interaction of these two types of path-dependency, due to stochastic habits and due to limited commitment, introduces considerations new to the literature.[3] Using the total derivative formula, we derive a drift and diffusion decomposition for optimal policy. We show that optimal policy dynamics change in the presence of participation constraints through an additional stochastic discount factor. This discount factor affects future marginal effects of habits by placing a larger weight on low marginal utility states.

Third, we analyze dynamic private information settings (Golosov, Kocherlakota, and Tsyvinski, 2003; Farhi and Werning, 2013; Kapička, 2013; Pavan, Segal, and Toikka, 2014; Golosov, Troshkin, and Tsyvinski, 2016). We first establish a new representation of incentive constraints using Malliavin integration by parts. This enables direct analysis of private information problems, even when a recursive representation is not feasible. With private information, we show that the planner reweights agents' utilities with a martingale weight. This weight is a stochastic integral over multipliers on prior incentive constraints along the trajectory of the skill shocks. This representation highlights a key economic difference between private information models and limited commitment models. In private information models, the weight is a martingale. In limited commitment models, the weight instead accumulates multipliers on previous participation constraints over time and, hence, only has a drift component (Marcet and Marimon, 2019).

We derive two new results on taxation in this dynamic private information economy. The first result gives the dynamics of the term structure of intertemporal distortions which describes how distortions vary with investment horizons and informs the difference in optimal capital taxes between short and long-term investments. We show that the term structure is determined by the product of the stochastic discount factor and the evolution of the martingale weight.

The second result derives the labor wedge when skills are an arbitrary function of the shock trajectory. Our general formulation of the skill process generalizes the results in the dynamic optimal taxation literature to incorporate recent labor economics evidence that focuses on earnings

[3] This combination may be particularly relevant for asset pricing as it jointly considers predictions of habit models (Abel, 1990; Constantinides, 1990; Campbell and Cochrane, 1999) and limited commitment models (Alvarez and Jermann, 2000).

processes with path dependency, nonlinearity of persistence, and realized skewness and kurtosis (Arellano, Blundell, and Bonhomme, 2017; Guvenen, Karahan, Ozkan, and Song, 2021; Browning, Ejrnæs, and Alvarez, 2010). The labor wedge is given by the labor supply elasticity and the product of two terms. The first term is the sensitivity of the stochastic discount factor to uncertainty. The second is the Malliavin semi-elasticity of skills which shows how path-dependent skills react to a shock in prior uncertainty. We derive a closed-form expression for this Malliavin semi-elasticity which governs the informational advantage of a misreporting agent over the planner. For path-dependent processes, the labor wedge significantly differs from the cases of the geometric Brownian motion and the Ornstein-Uhlenbeck process in the literature.

2 Environment

We consider a representative agent problem in continuous time with time $t \in [0, T]$. We denote consumption by c and habit consumption by x. Both consumption goods are produced with labor. Hours worked are given by n.

Production is linear in hours, with total output given by θn, where θ denotes labor productivity. Output can be used for consumption and habit consumption with a unit rate of transformation and can be transferred across periods and states at a unit rate. The feasibility constraint is:

$$E \int_0^T (c_t + x_t - \theta_t n_t) dt \leq 0. \tag{1}$$

The agent's lifetime utility of consumption and hours is given by:

$$E \int_0^T (u(c_t) + u(x_t) - h_t - v(n_t)) dt, \tag{2}$$

where utility from consumption at time t is $u(c_t)$, utility from habit consumption is $u(x_t) - h_t$, and labor disutility is $v(n_t)$. The functions u and $-v$ are strictly concave and satisfy the Inada conditions.[4] The period utility is $\mathcal{U}_t = u(c_t) + u(x_t) - h_t - v(n_t)$. Stochasticity in the economy is given by a Brownian motion w which affects both skills and habits. Skills depend on the trajectory w^t of the shocks between time 0 and t as $\theta_t(w^t)$.

Habits h_t depend on the trajectories of habit consumption x^t and shocks w^t as $h_t(x^t, w^t)$. Habits are linearly separable in past habit consumption:

$$h_t(x^t, w^t) = \int_0^t k_t^s(w^t) x_s ds, \tag{3}$$

[4] Time is finite and the discount rate is zero, both of which are non-essential.

where $k_t^s(w^t)$ are functions that depend on the shocks up to time t, w^t.[5] We highlight that habit coefficients depend both on the time of prior consumption s and the time of habit evaluation t,[6] and also depends on the shock trajectory w^t.

Discussion. The classic work of Stigler and Becker (1977) specifies utility depending on past consumption. Our environment is more general since habits are stochastic as they directly depend on the trajectory of past shocks. Consider a descriptive example. Stigler and Becker (1977) allow the appreciation of music to depend on past instances devoted to listening to music x^t. We further allow the taste for music to depend directly on personal experiences w^t.[7] A specific example of stochastic habits is Chetty and Szeidl (2016) who show that a heterogeneous agent model with consumption commitments behaves as a representative agent economy with stochastic habits in terms of aggregates. In their model, the habit coefficients on past consumption are generally stochastic and depend on the cross-sectional distribution of agents in the economy.[8]

While we frame our model as a habit economy, habits can be considered as general dependence of the economy on past shocks and decisions. Our specification specifically allows habits to depend on both past shocks and choices, which we refer to as stochastic hysteresis. We next present how hysteresis can be introduced in three important settings: macro-climate models, firm investment models, and knowledge accumulation models.[9]

First, our framework expands on dynamic macro-climate economies (Nordhaus, 1992, 1993) by incorporating path-dependent climate externalities, where h captures damages caused by past consumption of pollutant-emitting good x. Allowing for path-dependency is important, as there is significant evidence suggesting that important climate variables, such as vegetation, ice sheets, and ocean acidification, exhibit hysteresis.[10]

[5] We use habit specification (3) to clarify the exposition of results. Our insights extend to general habits $h_t(x^t, w^t)$ (see Riabov and Tsyvinski (2021)).

[6] For example, distant periods may be downweighted while recent periods are more salient as in the case of $k_t^s = e^{-\delta(t-s)}$, where $\delta > 0$ (Pollak, 1970; Ryder and Heal, 1973; Becker and Murphy, 1988; Constantinides, 1990).

[7] The appreciation for the Beatles among the current generation may grow by consuming their albums similar to their grandparents' generation, which we think of as being represented by x^t. However, the current generation can never first see the Beatles live on the Ed Sullivan Show, or attend the concert at Shea Stadium at the backdrop of the Vietnam war. We consider this trajectory of uncertain events as w^t that also influences appreciation of music.

[8] Our formulation is consistent with Frick, Iijima, and Strzalecki (2019) which axiomatizes general non-parametric dynamic random utility models and include both dependence on the history of shocks and on past choices, both of which are central to our specification of stochastic hysteresis.

[9] In Appendix A.1, we formally demonstrate the mapping between the economy in this section and these models.

[10] For example, the Intergovernmental Panel on Climate Change documents that a number of important climate variables show significant hysteresis behavior in their Fifth Assessment Report (Collins, Knutti, Arblaster, Dufresne, Fichefet, Friedlingstein, Gao, Gutowski, Johns, Krinner, Shongwe, Tebaldi, Weaver, and Wehner, 2013). Riabov

Second, Arrow (1964) and Rogerson (2008) analyze firm investment decisions with arbitrary deterministic depreciation patterns. Our generalization allows productive capital to be a general function of past investments and shocks. For example, productive capital can be a linear and separable function of past investments (3), where the current production value of past investments is given by $k_t^s(w^t)$. This permits deprecation to not only follow an arbitrary deterministic depreciation pattern but also to vary generally with depreciation shocks. For example, we allow for depreciation shocks that impact older machines but do not impact recent vintages.

Third, a large literature studies knowledge accumulation through both financial and time investments. Knowledge capital models endogenize labor productivity through past investment choices (Becker, 1962; Ben-Porath, 1967; Heckman, 1976), or with learning-by-doing (Arrow, 1962; Uzawa, 1965; Lucas, 1988). Under our generalization, knowledge capital is a general function (3) of past choices and stochastic realizations. The return on investments in education may depend on an individual's current age t, their age at the time of training s, and the shock trajectory. For example, past investments increase current productivity when they are more recent (t close to s), and when the training is more transferable, that is, an individual's current job is similar to their job at the time of their training (w^s close to w_t).[11]

3 Optimal Policy

The planner chooses consumption c_t, habit consumption x_t and hours n_t to maximize welfare (2) with habits (3) subject to the resource constraint (1). The optimality conditions are:

$$\lambda = u'(x_t) - E_t \int_t^T k_s^t\left(w^s\right) ds, \qquad \lambda = u'(c_t), \qquad \text{and} \qquad \lambda\theta_t(w^t) = v'(n_t), \tag{4}$$

where λ denotes the multiplier on the feasibility constraint (1). An increase in habit consumption increases current consumption utility and future habits. The effect on future habits is given by the conditional expectation $E_t \int_t^T k_s^t(w^s)ds$. Each coefficient k_s^t captures the effect of current habit consumption x_t on habits at future time s. This effect depends on the trajectory of shocks w^s as habits are stochastic. The future is uncertain from the perspective of time t which is captured by the conditional expectation of these future effects. The optimality conditions for consumption and labor are standard.

and Tsyvinski (2021) focus on macro-climate economies as in Golosov, Hassler, Krusell, and Tsyvinski (2014) and provide an extensive discussion of evidence on hysteresis in climate variables.

[11] Our notion of stochastic habits with unfamiliarity in Section 3.2 is an example of such specification.

The habit consumption wedge Λ_t is:

$$\Lambda_t := u'(x_t) - u'(c_t) = E_t \int_t^T k_s^t(w^s)ds. \tag{5}$$

The habit wedge plays the central role in habit economies. In Ljungqvist and Uhlig (2000), for example, it determines the tax that a government uses to align the choices of individuals with the social optimum in the presence of external habit formation. More broadly, the habit wedge corrects the marginal externality of habit consumption and is the optimal policy in models with path-dependent externalities.

Our primary objective is to describe the dynamics of optimal policy Λ_t. The main challenge in deriving the evolution of the habit consumption wedge lies in the fact that the conditional expectation $E_t \int_t^T k_s^t(w^s)ds$ is generally path-dependent, as the coefficients $k_s^t(w^s)$ are path-dependent. We thus cannot use the standard Ito formula as it applies only to functions of the state and not to functions of a path.

3.1 Optimal Policy Dynamics

We next develop the main theoretical tool of our paper, the total derivative formula for conditional expectation processes, to derive the drift and diffusion decomposition of optimal policy in closed form. These conditional expectations are central in dynamic economic models, as they reflect the optimal choices made by forward-looking agents who balance current costs and benefits against those in an uncertain future.

A preliminary step in developing the total derivative formula is to introduce the Malliavin derivative. This derivative characterizes how a function, which depends on the trajectory of a stochastic process, responds to small changes in that process and can thus be thought of as its sensitivity to shocks. Consider a variation of the path w^t that changes the drift of the stochastic process at each time $s \leq t$ by $\varepsilon \int_0^s z_r(w^r)dr$, where ε is small. The resulting change in the function value $F(w^t)$ can be represented, to a first order, as:

$$F\Big(w^t + \varepsilon \int_0^{\cdot} z_r dr\Big) = F(w^t) + \varepsilon \int_0^T D_r F(w^t) z_r dr + o(\varepsilon), \tag{6}$$

where $D_r F(w^t)$ is the Malliavin derivative at time r of the function $F(w^t)$.[12] Figure 1 illustrates

[12]Consider the Brownian motion $w_t = \int_0^t 1 dw_s$. The variation (6) is given by $w_t + \varepsilon \int_0^t z_r dr - w_t = \varepsilon \int_0^t z_r dr$. The Malliavin derivative $D_r F(w^t)$ is equal to one for all time periods $r \leq t$, and zero for all time periods $r > t$. In In Appendix A.2, we provide additional examples of Malliavin derivatives that we use in this paper. The notation $\int_0^{\cdot} z_r dr$ indicates that the entire trajectory is perturbed.

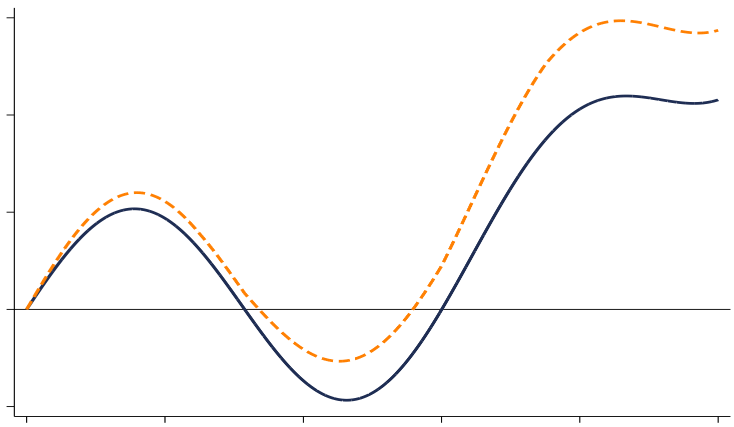

Figure 1: The Malliavin Derivative

Figure 1 illustrates the Malliavin derivative. The blue solid line shows trajectory w^t, while the orange dashed line shows a perturbed trajectory $w^t + \varepsilon \int_0^{\cdot} z_r dr$ which equals $w_t + \varepsilon \int_0^t z_r dr$ for all t. The Malliavin derivate evaluates the change in the function $F(w^t)$ due to a perturbation z_r at time r.

the Malliavin derivative, where the blue solid line shows trajectory w^t, while the orange dashed line shows a perturbed trajectory $w^t + \varepsilon \int_0^{\cdot} z_r dr$

The total derivative formula gives the drift and diffusion decomposition for the conditional expectation process.

Proposition 1. *Total Derivative Formula.* Let Λ_t be a process so that $\Lambda_t = E_t[\xi_t]$, where ξ_t is Malliavin differentiable and differentiable with respect to time. Then,

$$d\Lambda_t = E_t[\partial_t \xi_t]dt + E_t[D_t \xi_t]dw_t. \tag{7}$$

The proof is in Appendix A.3.[13]

The first term in the total derivative formula, $E_t[\partial_t \xi_t]$, gives the drift of the process and represents how the conditional expectation evolves with respect to time. The second term, $E_t[D_t \xi_t]$, gives the diffusion coefficient and represents how the conditional expectation evolves with changes in the underlying stochastic process.[14]

We apply the total derivative formula to optimal policy (5) to explicitly derive its drift and

[13]In Appendix A.3 we also prove a general version of the total derivative formula which does not require Malliavin differentiability. In Appendix A.4, we show that the assumption of time differentiability of ξ_t can be significantly relaxed, and note that when ξ_t does not vary with time, the Clark-Ocone formula is a special case of our formula.

[14]This explains why we call (7) the total derivative formula. For a function of two variables, the total derivative formula describes the function's change as the sum of the derivatives of each variable, multiplied by the respective changes in those variables. In our setting, the total derivative formula decomposes the change $d\Lambda_t$ as a combination of the changes dt and dw_t respectively weighted by the time derivative and the Malliavin derivative.

diffusion decomposition:

$$d\Lambda_t = \left(E_t \int_t^T \partial_t k_s^t(w^s)ds - k_t^t(w^t)\right)dt + \left(E_t \int_t^T D_t k_s^t(w^s)ds\right)dw_t. \tag{8}$$

The drift of the habit wedge is given by the time derivative of the effects of current consumption on future habits $E_t\partial_t \int_t^T k_s^t(w^s)$, or how expected marginal habits change with time. The stochasticity of habits translates into the diffusion coefficient that shows how expected habits evolve with respect to underlying stochasticity and is given by the expectation over Malliavin derivatives $E_t \int_t^T D_t k_s^t(w^s)ds$. The drift and diffusion decomposition thus breaks down complex interactions of time and uncertainty in the conditional expectation into a drift and a diffusion. The drift gives the expectation of optimal policy changes, while the diffusion gives its volatility. Given an initial condition, the decomposition then fully describes optimal policy along any trajectory.[15]

3.2 Examples: Unfamiliarity and Tipping Points

In this section, we solve two examples in closed form to illustrate the use of the total derivative formula (7).

Unfamiliarity. In the first example we capture the stochastic relevance of past consumption by introducing a notion of experiencing something different from what you were used to. An experience at time t is considered different from what you were used to by time s if the current realization $\log\theta_t$ differs from average productivity up to time s, $\log\bar{\theta}_s = \frac{1}{s}\int_0^s \log\theta_r dr$. The absolute value of the distance $\upsilon_s(\theta^t) := \log\theta_t - \log\bar{\theta}_s$ is a measure of unfamiliarity. Specifically, let:

$$h_t(x^t,\theta^t) = \alpha \int_0^t \exp(-(\delta + \kappa|\upsilon_s(\theta^t)|)(t-s))x_s ds, \tag{9}$$

where $\alpha \geq 0$ parameterizes the importance of habits. When $\kappa = 0$, we obtain the usual geometric habit specification that discounts experiences at more distant times in the past. When $\kappa > 0$, the weight on past consumption x_s is additionally decreased by $\kappa|\upsilon_s(\theta^t)|$ depending on how different $\log\theta_t$ is from what you were used to. This specification is an example of habits (3) where $k_t^s = \alpha\exp(-(\delta + \kappa|\upsilon_s(\theta^t)|)(t-s))$.[16]

[15] In the nested case with time-invariant habit coefficients $k_s(w^s)$, where $\partial_t k_s^t(w^s) = 0$, the drift of the marginal habit effect in (8) is $-k_t(w^t)$ (Detemple and Zapatero, 1991). When the habit coefficients k_s^t are deterministic, the Malliavin derivative $D_t k_s^t$ in equation (8) is zero. The time derivative, measuring how the future habits change throughout time, simplifies to $\int_t^T \partial_t k_s^t ds - k_t$.

[16] This formulation of habits captures three key features of how past experiences affect economic decisions outlined

The habit wedge Λ_t is a combination of the geometric habit wedge and the unfamiliarity term with strength κ. We view the habit consumption wedge as a portfolio of assets that pays an amount $\alpha \exp(-\delta(s-t)) \exp(-\kappa|\upsilon_t(\theta^s)|(s-t))$ for each history θ^s. Each component of the portfolio has a value determined by the discount rate δ and the additional path-dependent component determined by the absolute value of unfamiliarity $|\upsilon_t(\theta^s)|$.

The total derivative formula (7) describes the dynamics of optimal policy in closed form. The drift coefficient is $E_t \int_t^T \partial_t k_s^t ds - k_t^t$, where $\partial_t k_s^t = k_s^t\big(\delta + \kappa\,|\upsilon_t(\theta^s)| - \kappa(s-t)\partial_t \upsilon_t(\theta^s)\text{sign}(\upsilon_t(\theta^s))\big)$ and $k_t^t = \alpha$. The evolution with respect to time of the path-dependent unfamiliarity term $\upsilon_t(\theta^s)$ is given by $\partial_t \upsilon_t(\theta^s) = -\frac{1}{t}\upsilon_t(\theta^t)$. When the time horizon increases and the last realization exceeds the average, the average increases by $\frac{1}{t}\upsilon_t(\theta^t)$, and unfamiliarity $\upsilon_t(\theta^s)$ decreases by $\frac{1}{t}\upsilon_t(\theta^t)$. The diffusion term is $E_t \int_t^T D_t k_s^t(w^s) ds$, where $D_t k_s^t = -k_s^t \kappa(s-t)\text{sign}(\upsilon_t(\theta^s)) D_t \upsilon_t(\theta^s)$. The sensitivity to changes in the underlying path of uncertainty of the path-dependent unfamiliarity term $\upsilon_t(\theta^s)$ is the Malliavin semi-elasticity of the productivity process $D_t \upsilon_t(\theta^s) = D_t \log \theta_s = \frac{D_t \theta_s}{\theta_s}$. This semi-elasticity is the percentage change in productivity θ_s with respect to a change in the underlying stochastic process at time t.[17]

Tipping Point. The second example illustrates optimal policy in an important class of problems with tipping points. Tipping points are endogenous thresholds in time upon which the behavior of a system fundamentally changes.[18]

Specifically, we allow habits to significantly change once a new maximum in the trajectory of shocks is attained. Intuitively, this notion of a tipping point corresponds to the (best) "time of your life". Formally, the tipping point $\gamma_t(w^t) := \arg\max_{s\in[0,t]} w_s$ is the time when the trajectory w^t attains its maximum before age t, with the corresponding value $M_{0,t}(w^t) := \max_{s\in[0,t]} w_s$. Consider habit (3) with coefficients $k_t^s(w^t) = f(s - \gamma_t(w^t))$, where $f(x) = 0$ for all $x \le 0$. The behavior of the system changes as habits put zero weight on experiences prior to the tipping point $\gamma_t(w^t)$, and

in the review of Malmendier and Wachter (2022): long lasting effects, recency bias, and context dependence (Mullainathan, 2002; Malmendier and Nagel, 2011, 2016; Bordalo, Gennaioli, and Shleifer, 2020; Wachter and Kahana, 2021). A comparison with previous experiences as a similarity function is axiomatized in Billot, Gilboa, Samet, and Schmeidler (2005) and its exponential form in Billot, Gilboa, and Schmeidler (2008).

[17] In Appendix A.5, we explicitly calculate the conditional expectations in the drift and diffusion coefficients when the productivity process follows an Ornstein-Uhlenbeck process in logarithms.

[18] Tipping points are especially significant in models of climate change. For example, Dietz, Rising, Stoerk, and Wagner (2021) consolidate various approaches in the literature and argue for the need for models that incorporate sophisticated tipping point dynamics. Lemoine and Traeger (2014), Lontzek, Cai, Judd, and Lenton (2015), van der Ploeg and de Zeeuw (2018), and Cai and Lontzek (2019) incorporate tipping points in economics models of climate.

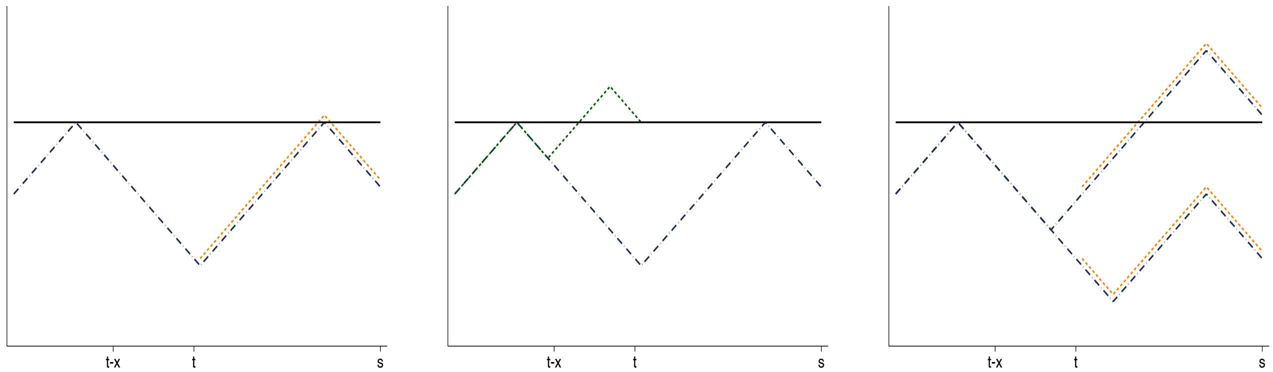

Figure 2: Tipping Points

Figure 2 shows how a perturbation of the process at time t can put the best time of your life up to time s after time $t-x$. In the left panel, the increase at time t puts the time of the maximum up to time s after $t-x$ as indicated by the orange dashed line exceeding the black solid line that indicates the maximum prior to the perturbation. The middle panel shows that when the maximum is already after time $t-x$ as indicated by the green line, a perturbation at time t does not change that. The right panel shows that a change at time t does not alter the indicator function $\mathbb{I}_{\{\gamma_s \leq t-x\}}$ if $M_{t,s} > M_{0,t}$ or if $M_{t,s} < M_{0,t}$.

agents form habits as $f(s-\gamma_t(w^t))$ only from this reference point onward.[19] The habit wedge is therefore given by $\Lambda_t = E_t \xi_t$ with $\xi_t = \int_t^T f(t-\gamma_s(w^s))ds$.

The total derivative formula gives the dynamics of optimal policy in closed form. The drift coefficient is $E_t \partial_t \xi_t = -f(t-\gamma_t) + E_t \int_t^T f'(t-\gamma_s)ds$. As time advances, future impacts on habits are amended to the extent that more time $t-\gamma_s$ has passed since the best time in life prior to time s. The term $-f(t-\gamma_t)$ measures the instantaneous decay of the habit externality with time.[20]

The diffusion coefficient captures how a perturbation in the underlying stochastic process at time t affects the future times of a maximum. A change in the underlying process puts the best time of your life up to time s, γ_s, after time $t-x$, where $x \geq 0$ is any amount of time, when two conditions are met. First, the time of your life before time t was before time $t-x$, or $\gamma_t \leq t-x$. Second, the maximum after time t has to be such that a perturbation at time t puts the best time of your life after time t (and hence after $t-x$). That is, prior to a perturbation the attained maximum values are identical, $M_{t,s} = M_{0,t}$, where $M_{t,s}$ is the maximum value attained between period t and s, or $M_{t,s}(w^s) := \max\limits_{r\in[t,s]} w_r$.[21]

[19]For example, at age 20, an individual's high point may have been being the high school valedictorian at age 18. An individual's habits reset after this achievement, meaning that only times $s \geq 18$ matter for habit formation. If this individual wins an Olympic gold medal at age 25, the weighting again resets, and only times $s \geq 25$ matter for habits. This specification immediately extends to habit coefficients that vary with time and habit coefficients that weight experiences prior to the reference point γ_t as well as experiences after the reference point.

[20]In Appendix A.6, we calculate $E_t \int_t^T f'(t-\gamma_s)ds$ explicitly.

[21]In summary, the indicator $\mathbb{I}_{\{\gamma_s \leq t-x\}}$ switches from one to zero if and only if $\mathbb{I}_{\{\gamma_t \leq t-x\}} \mathbb{I}_{\{M_{t,s}=M_{0,t}\}}$, and hence

The left panel of Figure 2 illustrates how a change in the process at time t can put the best time of your life up to time s after time $t-x$. The increase at time t puts the time of the maximum up to time s after $t-x$ as indicated by the orange dashed line exceeding the black solid line that indicates the maximum prior to the perturbation. The middle panel shows the necessity of the first condition that the time of your life before time t was before time $t-x$, or $\gamma_t \leq t-x$. If the maximum is already after time $t-x$ as indicated by the green line, then a perturbation at time t does not change that. The right panel shows the necessity of the second condition, $M_{t,s} = M_{0,t}$, by showing that a change at time t does not alter the indicator function if the maximum between t and s exceeds the maximum from 0 to t, $M_{t,s} > M_{0,t}$, or if $M_{t,s} < M_{0,t}$.

The optimal policy $\Lambda_t = E_t\xi_t = E_t \int_t^T \int_0^t f'(x)\mathbb{I}_{\{x\leq t-\gamma_s\}}dxds = \int_t^T \int_0^t f'(x)E_t\left[\mathbb{I}_{\{\gamma_s\leq t-x\}}\right]dxds$ highlights the importance of determining how a change in the stochastic process at time t affects the indicator $\mathbb{I}_{\{\gamma_s\leq t-x\}}$. The diffusion coefficient explicitly shows how a change in the trajectory of the stochastic process affects the likelihood of crossing an endogenous tipping point. We show that the diffusion coefficient is given in the closed form as $-f(t-\gamma_t)\int_t^T \varphi_{s-t}(M_{0,t}-w_t)ds$, where φ_{s-t} is the density function for the maximum of a Brownian motion over $s-t$ periods.[22]

4 Quantifying Optimal Policy

We build on our theoretical framework to quantitatively analyze optimal policy with stochastic hysteresis under unfamiliarity and compare our findings to those of Ljungqvist and Uhlig (2000).

We consider an infinite horizon economy ($T=\infty$) and assume there is only a habit good x. We replace the present value resource constraint (1) by resource constraints for every shock history θ^t such that $x_t(\theta^t) \leq \theta_t n_t(\theta^t)$. Consumption utility is $u(x) = \frac{x^{1-\gamma}-1}{1-\gamma}$, where $\gamma = 1.5$ and the disutility from work is $v(n) = n$. Logarithmic productivity follows an Ornstein-Uhlenbeck process for which the equivalent annual discrete-time AR(1) process has persistence equal to 0.9 and the standard deviation of the innovation equals 0.04.

We specify the habit coefficients as in the case of unfamiliarity (9). We parameterize the habits with $\alpha = 0.015$ and $\delta = 0.03$. In the case of geometric habits, $\kappa = 0$, optimal policy is given by

$\mathbb{I}_{\{\gamma_t\leq t-x\}}E_t\left[\mathbb{I}_{\{M_{t,s}=M_{0,t}\}}\right]$ in expectation.

[22]The conditional expectation that $M_{t,s}$ is equal to $M_{0,t}$ is $E_t[\mathbb{I}_{\{M_{t,s}=M_{0,t}\}}] = \varphi_{s-t}(M_{0,t}-w_t)$, where $\varphi_{s-t}(z) = \sqrt{\frac{2}{\pi(s-t)}}e^{-\frac{z^2}{2(s-t)}}$ is the density function for the maximum of a Brownian motion over $s-t$ periods. The diffusion coefficient is then $\int_t^T\int_0^t f'(x)\mathbb{I}_{\{\gamma_t\leq t-x\}}\varphi_{s-t}(M_{0,t}-w_t)dxds = -f(t-\gamma_t)\int_t^T \varphi_{s-t}(M_{0,t}-w_t)ds$.

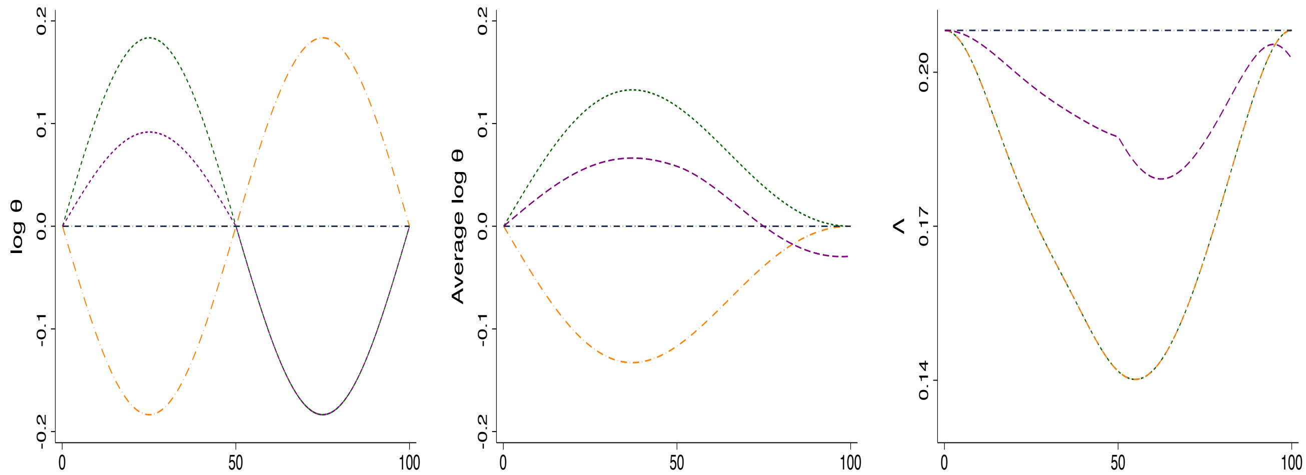

Figure 3: Optimal Policy with Stochastic Habits

Figure 3 shows optimal policy with stochastic habits described by unfamiliarity (9). The left panel displays four productivity paths, the middle panel plots the running average of productivity which determines how productivity paths translate into unfamiliarity $\upsilon_t(\theta^s) = |\log\theta_s - \log\bar{\theta}_t|$, and the right panel shows optimal policy. The figure shows that optimal policy is neither procyclical nor countercyclical as optimal policy over the green dashed expansion-contraction cycle is identical to optimal policy over the orange dashed contraction-expansion cycle.

$\Lambda_t = \alpha/\delta = 0.5$. An extra unit of consumption increases habits in all subsequent periods, and when discounted, the increase in future habit is given by $1/\delta$.

Optimal Policy. In order to demonstrate the optimal policy with stochastic habits (9), we present four sample paths for productivity and the corresponding optimal policy in Figure 3.

The left panel of Figure 3 displays four productivity paths. The blue dash-dotted line is a constant productivity path. The three business cycle lines are as follows. The green dashed line illustrates an expansion-contraction cycle, where productivity expands with a peak at period 25 and then enters a recession at period 50 with a trough at period 75. The orange dashed line is the mirror image of the green cycle in terms of cyclicality, resulting in a contraction-expansion cycle of the same magnitude. Consequently, differences between the green and orange paths are due to their opposing cyclicality. The purple dashed line portrays an asymmetric cycle that follows the same expansion-contraction pattern as the green line but with a muted expansion in the first half of the sample. Discrepancies in the first half of the sample are driven by varying magnitudes, while discrepancies in the second half arise from distinct trajectories in the first half and not from contemporaneous productivity differences. The middle panel plots the running average $\log\bar{\theta}_t := \frac{1}{t}\int_0^t \log\theta_r dr$ which determines how productivity paths translate into unfamiliarity

$\upsilon_t(\theta^s) = |\log\theta_s - \log\bar{\theta}_t|$. We now use the analytical characterization in Section 3.2 to generate the evolution of optimal policy along these trajectories and compare to the optimal policy under geometric habits $\Lambda_t = \alpha/\delta = 0.5$.

The first result is that optimal policy is neither procyclical nor countercyclical. The right panel of Figure 3 demonstrates that optimal policy over the green dashed cycle is identical to optimal policy over the orange dashed cycle. Since the unfamiliarity metric is a distance between current productivity and what you were used to at time t, $|\log\theta_s - \log\bar{\theta}_t|$, and since the green and orange cycle only differ through reflection, optimal policy is identical. In sum, the right panel of Figure 3 shows that optimal policy behaves identically in good and bad times as both unfamiliarity and the skill process are symmetric. This result stands in contrast to Ljungqvist and Uhlig (2000) where the habit wedge is instead countercyclical.[23] In Ljungqvist and Uhlig (2000) optimal policy Λ_t is proportional to the expected marginal utility of future habit consumption. In good times, the marginal utility of future habit consumption decreases since consumption increases with good shocks and shocks are persistent. Individuals thus relatively over-consume less in good times as the expected marginal utility of consumption decreases, and relatively over-consume more in bad times. In our model, good and bad times are equally unfamiliar and, hence, agents over-consume less in both good and bad times in the same relation compared to normal times.

The second point we emphasize is the lasting dependence of optimal policy on past shocks, or stochastic hysteresis. Stochastic hysteresis is illustrated by the difference between optimal policy under the green short-dashed expansion-contraction cycle and the purple dashed asymmetric cycle. While current productivity and expected productivity going forward in the second half of the sample are identical between the two paths, optimal policy differs owing to the dependence on past shocks. This dependence is strongly persistent. Even when the past 50 time periods are identical in terms of productivity, optimal policy differs between these two trajectories. We highlight that these paths also differ qualitatively. Toward the end of the sample, the habit wedge starts decreasing under the purple asymmetric cycle but continues increasing under the green expansion-contraction cycle. By the drift and diffusion decomposition (8), this is because the diffusion coefficient turns negative under the purple asymmetric cycle. While the shock increments are positive and identical

[23] Ljungqvist and Uhlig (2000) analyze a slightly different object than our habit wedge Λ_t. They focus on the linear income tax $\tau(\theta^t)$ that corrects the habit wedge. The relationship between the two is given by $\tau(\theta^t)/(1-\tau(\theta^t)) \propto \Lambda_t(\theta^t)\theta_t$. The habit wedge Λ_t in their model is countercyclical while the linear habit tax is procyclical due to productivity variation exceeding the cyclical variation in Λ_t.

to those under the green expansion-contraction cycle, the negative diffusion coefficient generates a decrease in optimal policy.

The third point is that, even when productivity follows a constant path, optimal policy is lower than under deterministic habits. Even if realized productivity is constant up to time t, the agent expects future productivity shocks to make today's experiences less relevant from the perspective of future habits. Therefore, today's consumption has a smaller effect on future habits compared to the geometric specification, where the effect of consumption on future habits is not affected by the business cycle trajectory.

Welfare. We show that unfamiliarity has significant welfare consequences. Let the consumption-equivalent welfare difference between the planner's outcome and the laissez-faire economy be Δ. Let Δ_κ denote the consumption-equivalent welfare difference between the planner outcome and the economy with policy at the optimal value under deterministic habits α/δ. The first measure captures the overall welfare gains while the second measure captures the welfare gains due to accounting for the stochastic nature of habits.[24]

Table 1 presents the welfare results for different parameterizations of unfamiliarity habits. The first column shows that under deterministic habits, the welfare gains of optimal policy increase with the lifetime relevance of deterministic habits α/δ. As α/δ increases from 0.2 to 0.5 and 0.8, the welfare gains of moving to optimal policy increase from 0.7 to 3.5 and 7.4 percent of optimal consumption.

The welfare gains of optimal policy are smaller in the presence of stochastic habits. Specifically, the second column of Table 1 shows that welfare gains of optimal policy are significantly smaller, by about two thirds, under stochastic habits. The optimal wedge is much smaller than α/δ because future stochastic outcomes make present habit consumption less relevant.

The third column evaluates the welfare gains of optimal policy in comparison to implementing the (suboptimal) optimal policy for deterministic habits α/δ. This column shows that welfare gains of optimal policy cannot be captured by optimal policy for deterministic habits, highlighting the relevance of accounting for the stochastic nature of habits in policy design.

[24]These comparisons are analogous to those in Ljungqvist and Uhlig (2000). Suppose the planner allocation is $\{x_t, n_t\}$ and the laissez-faire allocation is $\{\tilde{x}_t, \tilde{n}_t\}$. The consumption-equivalent welfare difference between the planning economy and laissez-faire economy Δ solves: $E\int_0^\infty (u((1-\Delta)x_t) - h_t - v(n_t))dt = E\int_0^\infty (u(\tilde{x}_t) - \tilde{h}_t - v(\tilde{n}_t))dt$. In Appendix A.7 we show that the results in this section are robust to a range of alternative parameterizations of the preferences and the productivity process.

Table 1: Welfare Loss Under Suboptimal Policy

α/δ	$\kappa = 0$	$\kappa = 1$	
	Δ	Δ	Δ_κ
0.2	0.7	0.4	0.2
0.5	3.5	1.6	1.2
0.8	7.4	3.1	2.6

Table 1 presents welfare results for different parameterizations of unfamiliarity habits. The second column shows the welfare gains of optimal policy with geometric habits ($\kappa = 0$), while the third column shows the welfare gains of optimal policy with stochastic habits ($\kappa = 1$). The fourth column evaluates the welfare gains of optimal policy with stochastic habits in comparison to implementing the (suboptimal) optimal policy for deterministic habits, and shows that welfare gains of optimal policy cannot be captured by optimal policy for deterministic habits.

5 Robustness

We use the methodology of Section 3 to develop the analysis of robust policy in the presence of stochastic hysteresis.[25] Robustness can be particularly important in our path-dependent framework, as habits (3) directly depend on the shock trajectory and are thus affected by misspecification.

5.1 Policy with Concerns for Robustness

We introduce concerns for robustness following Hansen and Sargent (2001, 2008) into our problem with stochastic hysteresis. The stochastic process can be misspecified so that it is difficult to distinguish from the true stochastic process. This is captured by the misspecified Brownian motion $w^h = w + \int_0^\cdot h ds$ in which the drift of history w^t is changed by $\int_0^t h_s(w^s)ds$.

The concern for robustness is represented by the optimization problem

$$\min_h \; E\big[\mathcal{U}(w^h)\big], \tag{10}$$

with lifetime utility $\mathcal{U}_t = \int_0^T \mathcal{U}_t dt$, and where misspecification is constrained by the relative entropy $R(\mu_h||\mu)$ between the distribution of the misspecified process μ_h and the true distribution of the Brownian motion μ: $R(\mu_h||\mu) \leq A$.

The problem for the choice of misspecification h can be represented as $\min\limits_h E_\mu[\mathcal{U}(w^h)] + \frac{1}{\kappa}R(\mu_h||\mu)$ where welfare is penalized by relative entropy with strength $1/\kappa$. By the variational

[25] The detailed derivations are in Appendix A.8.

formula for relative entropy, the value of the optimum for the misspecification problem is given by $-\frac{1}{\kappa}\log E\exp(-\kappa\mathcal{U}(w))$.

Therefore, concerns for robustness lead to the planner problem:

$$\max_{c,x,n} -\frac{1}{\kappa}\log E\exp(-\kappa\mathcal{U}(w)), \tag{11}$$

subject to the resource constraint (1). By maximizing the expectation of the exponentiated utility, the planner cares not only about expected welfare (2) but also about the variability of the welfare function.

We show that optimal robust policy Λ_t^R is:

$$\Lambda_t^R = \frac{E_t \int_t^T e^{-\kappa\mathcal{U}(w)} k_s^t ds}{E_t e^{-\kappa\mathcal{U}(w)}}. \tag{12}$$

Robustness considerations exponentially tilt the habit coefficients k_s^t by $\frac{e^{-\kappa\mathcal{U}(w)}}{E_t e^{-\kappa\mathcal{U}(w)}}$. This tilting puts more weight on trajectories associated with low lifetime utility and less weight on trajectories associated with high lifetime utility. In sum, optimal policy with concerns for robustness to model misspecification takes the same form as optimal policy without model misspecification (5) but with habit coefficients tilted towards low lifetime utility paths.

We next use the total derivative formula (7) to describe the dynamics of optimal robust policy. The numerator in Λ_t^R is $E_t \int_t^T \tilde{k}_s^t(w^s)\,ds$ where habit coefficients are modified in proportion to the exponential tilt $\tilde{k}_s^t(w^s) := k_s^t(w^s) E_s\left[e^{-\kappa\mathcal{U}(w)}\right]$. The application of the total derivative formula immediately gives its drift and diffusion decomposition:

$$d\left(E_t \int_t^T \tilde{k}_s^t(w^s)ds\right) = \left(E_t \int_t^T \partial_t \tilde{k}_s^t(w^s)ds - \tilde{k}_t^t(w^t)\right)dt + \left(E_t \int_t^T D_t \tilde{k}_s^t(w^s)ds\right)dw_t. \tag{13}$$

The dynamics of optimal policy are modified to account for exponential tilting. The drift reflects only the level adjustment as exponential tilting does not evolve with time: $\partial_t \tilde{k}_s^t(w^s) = E_s\left[e^{-\kappa\mathcal{U}(w)}\right]\partial_t k_s^t(w^s)$. The adjustment in the diffusion coefficient reflects not only sensitivity of future habits to underlying uncertainty adjusted in level by the tilt but also the sensitivity of the tilt itself $D_t\tilde{k}_s^t(w^s) = E_s\left[e^{-\kappa\mathcal{U}(w)}\right] D_t k_s^t + k_s^t E_s\left[D_t e^{-\kappa\mathcal{U}(w)}\right]$, where sensitivity is captured by the Malliavin derivative.[26] This result, which characterizes optimal policy dynamics in a general path-dependent environment, is new to the robustness literature that predominantly uses recursive representations.

[26]Since the denominator of optimal policy with robustness concerns Λ_t^R in (12) is a martingale, the dynamics $d\Lambda_t^R$ is given by Ito's lemma and presented in Appendix A.8.

5.2 Sensitivity of Policy to Misspecification

The second form of robustness we study is how optimal policy changes with local misspecification of the underlying process. This connects to robust statistics (Hampel, 1974; Huber, 1981), which studies how estimators change in response to misspecification of underlying data. We show that this local sensitivity of policy to misspecification can be analyzed using Malliavin derivatives. Our approach and characterization of first-order robustness of optimal policy and its dynamics using Malliavin derivative is new to the robustness literature.

We introduce local misspecification of the stochastic process by considering a perturbation as in Figure 1. The shift in optimal policy in response to such misspecification is represented by its Malliavin derivative as in (6):

$$D_r\Lambda_t = E_t \int_t^T D_r k_s^t(w^s)ds. \tag{14}$$

The sensitivity of optimal policy to misspecification is equal to the conditional expectation of the sensitivities of the habit coefficients. Therefore, optimal policy under misspecification Λ_t^M is,:

$$\Lambda_t^M = \Lambda_t + \varepsilon \int_0^t D_r\Lambda_t h_r dr + o(\varepsilon). \tag{15}$$

That is, optimal policy without misspecification Λ_t is adjusted by first-order effects of misspecification at all prior times r as measured by its Malliavin sensitivity $D_r\Lambda_t$ and its size h_r.

The sensitivity of the dynamics of optimal policy with misspecification is, up to first-order:

$$d\Lambda_t^M = d\Lambda_t + \varepsilon \int_0^t (dD_r\Lambda_t)h_r dr + \varepsilon D_t\Lambda_t h_t dt. \tag{16}$$

That is, the dynamics of optimal policy $d\Lambda_t$ in equation (8) is adjusted by two additional terms. The first term accounts for $dD_r\Lambda_t$ which are the evolutions of sensitivity of optimal policy to misspecification at times $r < t$ given by $D_r\Lambda_t$.[27] The second term accounts for $D_t\Lambda_t$, sensitivity of optimal policy Λ_t to misspecification at moment t.

[27] Specifically, $dD_r\Lambda_t = D_r\alpha_t dt + D_r\beta_t dw_t$, where α_t and β_t are the drift and diffusion coefficients of optimal policy $d\Lambda_t$ in equation (8). There is a parallel between (16) and the literature on sensitivity of asset prices to underlying parameters (Fournié, Lasry, Lebuchoux, Lions, and Touzi, 1999; Borovička, Hansen, Hendricks, and Scheinkman, 2011; Hansen, 2012; Hansen and Scheinkman, 2012; Borovička, Hansen, and Scheinkman, 2014; Borovička and Hansen, 2016). In these papers, shocks are driven by a diffusion $Y_t(w_t)$ and, therefore, do not feature direct path-dependency. The drift and diffusion coefficients then simplify to $D_r\alpha_t = \alpha'(Y_t)D_rY_t$ and $D_r\beta_t = \beta'(Y_t)D_rY_t$.

6 Limited Commitment

We incorporate limited commitment frictions (Thomas and Worrall, 1988; Kehoe and Levine, 1993; Kocherlakota, 1996; Marcet and Marimon, 2019) into the model with stochastic habits.[28] Limited commitment frictions introduce additional path-dependence through the participation constraints and interacts with path-dependency due to habits. Our main result is an explicit drift and diffusion decomposition of the evolution of the habit wedge in the limited commitment model. This result is new to the literature on limited commitment.

After each history w^r, the participation constraint requires that the individual's expected utility from staying in the relationship exceeds an outside option value:

$$E_r \int_r^T \Big(u(c_t) + u(x_t) - \int_0^t k_t^s x_s ds - v(n_t) \Big) dt \geq U_r(w^r). \tag{17}$$

where the outside option value $U_r(w^r)$ is a general function of the path w^r. The planner maximizes welfare (2) with habits (3) subject to the resource constraint (1) and participation constraints (17).

Additional utility from consumption, habit consumption, or leisure relaxes participation constraints (17) for all $0 \leq r \leq t$. The value of relaxing the participation constraint at time r is given by the multiplier μ_r. Habit consumption x_t relaxes prior participation constraints and also tightens future participation constraints. These forces are balanced in the optimality conditions for consumption, labor supply, and habit consumption:[29]

$$\frac{\lambda}{1 + \int_0^t \mu_r dr} = u'(c_t) = \frac{v'(n_t)}{\theta_t}, \qquad \text{and} \qquad u'(c_t) = u'(x_t) - E_t \int_t^T k_s^t m(c_s, c_t) ds. \tag{18}$$

where the stochastic discount factor $m(c_s, c_t) = \frac{u'(c_t)}{u'(c_s)} = \frac{1+\int_0^s \mu_r dr}{1+\int_0^t \mu_r dr}$. There is no distortion between labor supply and consumption since both consumption utility and labor disutility are equally reweighted by the cumulative multiplier $1 + \int_0^t \mu_r dr$.

Habit consumption affects habits in all future periods s which in turn affects the commitment constraints in all periods $0 \leq r \leq s$. The latter results in additional weight $\int_0^s \mu_r dr$ on the habit coefficient k_s^t. The habit wedge is:

$$\Lambda_t = E_t \int_t^T k_s^t m(c_s, c_t) ds. \tag{19}$$

[28] Krueger and Uhlig (2022) present an analytically tractable framework for the limited commitment models in neoclassical growth settings.

[29] The detailed derivations are in Appendix A.9.

One can intuitively think of optimal policy Λ_t as the price of an asset that pays dividends k_s^t. These dividends are valued by the stochastic discount factor $m(c_s, c_t)$ which is driven by the inverse marginal utility (Kocherlakota and Pistaferri, 2009).

We use the total derivative formula (8) to derive the evolution of optimal policy:[30]

$$d\Lambda_t = \Big(E_t \int_t^T \partial_t \left(k_s^t m(c_s, c_t)\right) ds - k_t^t\Big) dt + E_t\Big(\int_t^T D_t \left(k_s^t m(c_s, c_t)\right) ds\Big) dw_t \tag{20}$$

The drift term is analogous to the drift without participation constraints (8) but is now multiplied by the stochastic discount factor m. Without participation constraints the stochastic discount factor always equals one. With participation constraints, the stochastic discount factor is distorted by the stochastic weight $(1 + \int_0^s \mu_r dr)/(1 + \int_0^t \mu_r dr)$ and generally differs from one. This discount factor affects the future marginal effects of habits by putting more weight on low marginal utility states. The presence of the stochastic discount factor in the habit wedge (19) hence introduces additional drift and diffusion terms in the evolution of optimal policy with limited commitment.

7 Dynamic Private Information

We generalize the analysis of dynamic economies with private information (Golosov, Kocherlakota, and Tsyvinski, 2003; Farhi and Werning, 2013; Kapička, 2013; Pavan, Segal, and Toikka, 2014; Golosov, Troshkin, and Tsyvinski, 2016) by considering general path-dependent skill processes. We first derive a new non-recursive representation of the planning problem that enables direct analysis of dynamic private information economies, even when a recursive representation is not feasible. We then establish two new results on dynamic taxation – the term structure of the intertemporal distortion and the labor wedge with path-dependent skill processes.

Our main specification of the skill processes is

$$\log \theta_t = \int_0^t \mu_t^s(w^s) ds + \int_0^t \sigma_t^s(w^s) dw_s, \tag{21}$$

for which both the drift $\mu_t^s(w^s)$ and diffusion coefficients $\sigma_t^s(w^s)$ depend on the shock trajectory.

Recent developments in labor economics emphasize the rich nature of labor income shocks with path-dependency, nonlinearity of persistence and realized skewness and kurtosis (Arellano, Blundell, and Bonhomme, 2017; Guvenen, Karahan, Ozkan, and Song, 2021; Browning, Ejrnæs,

[30] This result assumes that $m(c_s, c_t)$ is Malliavin differentiable. The general version of the total derivative formula in Appendix A.3 does not require Malliavin differentiability.

and Alvarez, 2010).[31] Our specification features path dependency through the stochastic integral of past shocks and through path-dependence in coefficients as well as nonlinearity of persistence and realized skewness and kurtosis through nonlinearity in the skill process coefficients. When skills are driven by these processes the dynamic private information problem does not permit a recursive representation.

To highlight the results on path-dependent skills, we assume there is only a consumption good and no habit good.[32]

7.1 Incentive Constraints

Consider a problem where the planner observes only labor output $y_t = \theta_t n_t$ but not individual productivity or hours worked. Unobservability of shocks introduces incentive constraints into the planning problem. When an individual with a history of idiosyncratic shocks w^t (and skill $\theta_t(w^t)$) reports $\hat{w}^t$, it receives allocations $c_t(\hat{w}^t)$ and $y_t(\hat{w}^t)$. The incentive constraint implies that agents maximize utility by reporting their true type w, or $w \in \arg\max_{\hat{w}} \; E\int_0^T \left(u(c_t(\hat{w}^t)) - v\left(\frac{y_t(\hat{w}^t)}{\theta_t(w^t)}\right)\right)dt$, where $\hat{w}_t = w_t + \varepsilon \int_0^t z_r dr$ and z_r depends on the history up to period r. Intuitively, agents may misreport by changing the drift of the process w, as illustrated in Figure 1.[33]

We introduce Malliavin integration by parts, a stochastic analogue of the common integration by parts, to provide a new characterization of incentive constraints. For a Malliavin differentiable function $F(w^t)$:

$$E\Big[\int_0^t D_r F(w^t) z_r dr\Big] = E\Big[F(w^t)\int_0^t z_r dw_r\Big]. \tag{22}$$

[31] Arellano, Blundell, and Bonhomme (2017) develop a model with nonlinear income dynamics that is able to generate a general form of conditional heteroskedasticity, skewness, and kurtosis. They emphasize the nonlinear persistence of histories where the impact of past shocks on current earnings varies nonlinearly with new shocks. Their general specification also extends to the underlying processes being a higher-order Markov process for persistence and a moving average transitory component. Guvenen, Karahan, Ozkan, and Song (2021) document that earnings changes display strong negative skewness and high kurtosis. Browning, Ejrnæs, and Alvarez (2010) extend the conventional ARMA model to incorporate additional initial convergence to the long run process. Browning and Ejrnæs (2013) argue that the ARMA(1,2) model with significant heterogeneity in parameters and nonlinear trends fits the data and is consistent with a wide range of models. Altonji, Hynsjö, and Vidangos (2022) is a recent review of this literature.

[32] See Appendix A.10 for the derivations for the dynamic private information economy. We additionally provide a characterization of the optimum with both the consumption and the habit good. Boerma (2019) studies a dynamic private information economy where housing services play a role similar to our habit consumption.

[33] In Appendix A.11 we show that this misreporting is without loss of generality. Specifically, we show that any absolutely continuous change of measure can be represented by such misreports. The absolute continuity means the planner cannot detect these deviations as all events possible when agents tell the truth are possible when agents misreport.

Integration by parts is one of the cornerstones of Malliavin calculus. It states that the expectation of the Malliavin derivative $\int_0^t D_r F(w^t) z_r dr$ is equal to the expectation of the function $F(w^t)$ times the stochastic integral $\int_0^t z_r dw_r$.[34]

Using Malliavin differentiation and integration by parts, the local incentive constraints for a small ε are:

$$0 = E \int_0^T \mathcal{U}_t \Big(\int_0^t z_r dw_r \Big) dt - E \int_0^T v' \Big(\frac{y_t}{\theta_t} \Big) \frac{y_t}{\theta_t} \int_0^t \frac{D_r \theta_t}{\theta_t} z_r dr dt, \tag{23}$$

where $\mathcal{U}_t$ is the truth-telling utility at time t. This result gives a new, non-recursive representation of the incentive constraints.

The first term is the utility difference between truth-telling agents with shocks w and truth-telling agents with shocks $w + \varepsilon \int_0^{\cdot} z_r dr$. A small change in the shock trajectory in period r changes utility of an agent in period t by $D_r \mathcal{U}_t(w^t) z_r$ which is the sensitivity of utility $D_r \mathcal{U}_t(w^t)$, represented by the Malliavin derivative, times the sizes of the misreports z_r. The lifetime utility difference is then given by $\int_0^t D_r \mathcal{U}_t(w^t) z_r dr$. The key step is to use Malliavin integration by parts to write the change in lifetime utility as $E\big[\int_0^t D_r \mathcal{U}_t(w^t) z_r dr \big] = E\big[\mathcal{U}_t(w^t) \int_0^t z_r dw_r \big]$. This expression shows that the utility change of the misreporting agent is the truthful utility weighted by the stochastic integral $\int_0^t z_r dw_r$.

The second term is marginal disutility of effort with respect to productivity $v'(n_t) n_t$ times the sensitivity of productivity to the trajectory of shocks and the size of the misreport z_r, $\int_0^t \frac{D_r \theta_t}{\theta_t} z_r dr$. This additional information rent appears for labor but not for consumption because the marginal disutility of labor at time t directly depends on skill θ_t while the marginal disutility of consumption does not.

7.2 Optimal Policy

Optimal policy maximizes aggregate welfare (2) subject to the resource constraint $E \int_0^T (y_t - c_t) dt \geq 0$ and the local incentive constraints (23). This gives the Lagrangian:

$$\mathcal{L} = E \int_0^T \left(\mathcal{U}_t \Big(1 + \int_0^t \mu_r dw_r \Big) - v' \Big(\frac{y_t}{\theta_t} \Big) \frac{y_t}{\theta_t} \Big(\int_0^t \frac{D_r \theta_t}{\theta_t} \mu_r dr \Big) + \lambda (y_t - c_t) \right) dt. \tag{24}$$

[34] The intuition for Malliavin integration by parts can be based on the Girsanov theorem for the change of measure. The Girsanov theorem states that the expectation of the function of a Brownian motion and the expectation of that function with the drift changed by $Z_t = \varepsilon \int_0^t z_r dr$ relate through $E[F(w^t + \varepsilon \int_0^{\cdot} z_r dr) \mathcal{E}(Z_t)] = E[F(w^t)]$ where $\mathcal{E}(Z_t) = \exp(\varepsilon \int_0^t z_s dw_s - \frac{1}{2} \varepsilon^2 \int_0^t z_r^2 dr)$ is the stochastic exponential. Equation (22) is then obtained by evaluating the derivatives on both sides. The change of measure is thus the stochastic integral $\varepsilon \int_0^t z_r dw_r$ when ε is small.

where λ is the multiplier for the resource constraint, and μ_r in the multiplier for the incentive constraints.[35]

We highlight two key differences between the problem with private information (24) and the problem with limited commitment (18). First, while both models put additional weight on utility $\mathcal{U}_t$, the form of this additional weight is different. For limited commitment models, Marcet and Marimon (2019) show that the additional weight is an integral $\int_0^t \mu_r dr$ that accumulates random multiplier values over time. For private information models, we show that the additional weight is a stochastic integral $\int_0^t \mu_r dw_r$ where integration is over the stochastic trajectory. It thus accumulates both random multiplier values and random fluctuations of the stochastic process.[36]

The difference between these two integrals highlights the main differences between the limited commitment problem and the private information problem. The stochastic integral $m_t = 1 + \int_0^t \mu_r dw_r$ is a martingale with $dm_t = \mu_t dw_t$ and its evolution is driven by the diffusion term. On average, stochasticity does not affect agents but individual realizations do. The evolution of the martingale weight implies that the evolution of individual variables, such as the inverse marginal utility of consumption, is driven by random innovations dw_t. In limited commitment models, the evolution of the cumulative multiplier given by the random integral $g_t = 1 + \int_0^t \mu_r dr$ is $dg_t = \mu_t dt$ and the evolution of individual variables is instead driven by the drift term and is thus locally deterministic.

Second, models with private information have an additional term to account for the information advantage that individuals have over the planner represented by $\int_0^t \frac{D_r\theta_t}{\theta_t}\mu_r dr$. This term mirrors the cumulative multiplier in limited commitment models but weighted by the Malliavin semi-elasticity of productivity $\frac{D_r\theta_t}{\theta_t}$.

Term Structure for Intertemporal Wedges. The optimality condition for consumption is:

$$\lambda = u'(c_t)\left(1 + \int_0^t \mu_r dw_r\right). \tag{25}$$

The marginal resource cost λ is equated to the marginal utility of consumption and the marginal benefit of relaxing the incentive constraint $\int_0^t \mu_r dw_r$.[37]

[35] Since the incentive constraints (23) are linear in the perturbation of the trajectory, they enter the Lagrangian with perturbation z_r replaced by the multiplier μ_r. We present the detailed derivation in Appendix A.10.

[36] This reweighting is a form of stochastic habits where utility becomes $\mathcal{U}_t(w^t)(1 + \int_0^t \mu_r dw_r)$. This connection parallels the discussion in Chien and Lustig (2010) who in a limited commitment model argue that the reweighting is a form of a stochastic habit.

[37] The inverse Euler equation $\frac{1}{u'(c_t)} = E_t\left[\frac{1}{u'(c_{t+\delta})}\right]$ (Golosov, Kocherlakota, and Tsyvinski, 2003) follows by noting

We derive a result new to the literature − the term structure of the intertemporal distortion which describes how the distortion varies with the investment horizons δ. For example, the term structure of the intertemporal distortion determines the difference in optimal capital taxes on short versus long-term assets. We define the term structure of the intertemporal distortion between periods t and $t+\delta$ for $\delta > 0$ as:

$$\tau_t^\delta := 1 - \frac{E_t u'(c_{t+\delta})}{u'(c_t)}. \tag{26}$$

The instantaneous savings wedge is nested as $\lim_{\delta \to 0} \tau_t^\delta / \delta$ which describes a short term distortion.

In order to characterize the term structure of the intertemporal distortion we first analyze the stochastic discount factor. By the optimality condition for consumption (25), the stochastic discount factor is $m(c_t) = \frac{u'(c_0)}{u'(c_t)} = 1 + \int_0^t \mu_r dw_r$ and evolves as $dm(c_t) = \mu_t dw_t$. The evolution is driven by the incentive constraints represented by the multiplier μ_r and the underlying uncertainty dw_r. When the realization of uncertainty dw_r is positive, the stochastic discount factor increases, meaning that the consumption of the agent increases.

The magnitude of the evolution of the stochastic discount factor is governed by the multiplier μ_t. One advantage of our approach is that it enables a direct representation for the multipliers on the incentive constraints:

$$\mu_r = E_r\big[D_r m(c_t)\big], \tag{27}$$

where $r < t$. This expression gives a clear interpretation to the multiplier as the sensitivity of the stochastic discount factor to changes in the underlying uncertainty.

The term structure τ_t^δ sums the product of the stochastic discount factor $m(c_s, c_t) = \frac{m(c_s)}{m(c_t)} = \frac{1+\int_0^s \mu_r dw_r}{1+\int_0^t \mu_r dw_r}$ for $t \leq s$ and the squared sensitivity of the future marginal utility $\frac{E_s[D_s m(c_{t+\delta})]}{m(c_s)}$ for all dates between current time t and the investment horizon $t+\delta$:

$$\tau_t^\delta = -E_t \int_t^{t+\delta} m(c_t, c_s) \left(\frac{E_s[D_s m(c_{t+\delta})]}{m(c_s)} \right)^2 ds. \tag{28}$$

The squared term is a consequence of Ito's lemma and reflects the diffusion coefficient of the martingale process governing the inverse marginal utility of consumption. In Appendix A.10, we use the total derivative formula (7) to give the explicit drift and diffusion decomposition of the evolution of the term structure.

that the stochastic weight is a martingale.

Labor Wedge with Stochastic Hysteresis. Hours worked are distorted in two ways. The first is the same reweighting $1 + \int_0^t \mu_r dw_r$ of disutility from labor as for consumption and does not introduce a distortion in the tradeoff between hours and consumption. The second distortion is due to the additional informational advantage that an agent has over the planner in knowing their marginal disutility of work. This informational advantage results in a labor wedge, which is defined as $\tau_t^l = 1 - \frac{v'(n_t)\frac{1}{\theta_t}}{u'(c_t)}$, and that is optimally given by:

$$\frac{\tau_t^l}{1-\tau_t^l} = \varepsilon_t \int_0^t \frac{D_r\theta_t}{\theta_t} \frac{E_r[D_r m(c_t)]}{m(c_t)} dr, \tag{29}$$

where we used (27) for the multiplier μ_r.

Equation (29) is new to the literature and derives the labor wedge when skills are an arbitrary function of the trajectory of shocks, generalizing the dynamic labor wedge formulas in Farhi and Werning (2013) and Golosov, Troshkin, and Tsyvinski (2016). There are two terms in equation (29). The first term $\varepsilon_t = 1 + \frac{v''(n_t)n_t}{v'(n_t)}$ is the elasticity of labor supply. The planner wants to have low taxes on more elastic agents. The second term captures the information rents that an agent receives due to unobservability of the marginal disutility of labor. This term is the product of the sensitivity of the discount factor $m(c_t)$ to uncertainty at time r, represented by the Malliavin derivative $E_r\big[D_r m(c_t)\big]$ scaled by the stochastic discount factor $m(c_t)$ itself, and the Malliavin semi-elasticity of skills $\frac{D_r\theta_t}{\theta_t} = D_r \log\theta_t$.

We next derive a closed form expression for the Malliavin semi-elasticity of the general skill processes (21):

$$D_r \log\theta_t = \int_r^t D_r\mu_t^s(w^s)ds + \int_r^t D_r\sigma_t^s(w^s)dw_s + \sigma_t^r(w^r), \tag{30}$$

for $r < t$. The Malliavin semi-elasticity contains three terms: the Malliavin derivative of the drift $D_r\mu_t^s(w^s)$, the Malliavin derivative of the diffusion $D_r\sigma_t^s(w^s)$, and the term $\sigma_t^r(w^r)$ which can be thought of as a derivative with respect to the innovation dw_r.

As an example, consider the skills $\log\theta_t = \int_0^t k_t^s dw_s$. The productivity at time t is the weighted average of the innovations in the trajectory with weights k_t^s. The Malliavin semi-elasticity is $D_r \log\theta_t = k_t^r$ stating that the sensitivity to a shock at time r is varying with both current time t and the time of the shock r. The optimal labor wedge is: $\frac{\tau_t^l}{1-\tau_t^l} = \varepsilon_t \int_0^t k_t^r \frac{E_r[D_r m(c_t)]}{m(c_t)} dr$. The labor wedge can be seen as a stochastic habit where the planner incorporates incentive constraints at all prior times r with weights k_t^r, scaled by the labor supply elasticity ε_t. These weights represent the importance of past innovations to today's skill and govern the agent's informational advantage.

We now compare the labor wedge for path-dependent skills in equation (29) to the labor wedge when logarithmic skills are given by the Brownian motion with diffusion coefficient σ or by the Ornstein-Uhlenbeck process with drift $-\zeta \log \theta_t$ and diffusion σ. For the Brownian motion $\frac{D_r \theta_t}{\theta_t} = \sigma$ so the deviation at any time r has the same percentage effect σ on skill θ_t. The informational advantage of the agent fully persists and is captured by the adjustment equal to σ independent of the time of misreporting. For the Ornstein-Uhlenbeck process $\frac{D_r \theta_t}{\theta_t} = \sigma e^{-\zeta(t-r)}$. Since this process mean reverts, more distant periods r are downweighted exponentially compared to period t as the informational advantage of the misreporting agent dwindles with time which tends to decrease the labor wedge.[38] For the path-dependent process (21), the informational advantage is given by (30) which may significantly differ from a constant (as for the Brownian motion) and from a geometric decay (as for the Ornstein-Uhlenbeck process). For example, suppose early-life experiences play a more important role than later experiences (Shonkoff and Phillips, 2000; Knudsen, Heckman, Cameron, and Shonkoff, 2006; Heckman, 2006; Cunha, Heckman, and Schennach, 2010; Chetty, Hendren, and Katz, 2016), and k_t^r is high for formative years of life. The informational advantage that the agent derives from these years plays a disproportionate role which tends to increase the labor wedge compared to the cases of both the Brownian motion and the Ornstein-Uhlenbeck process.

Finally, consider an example where logarithmic productivity is the running maximum of the trajectory $\log \theta_t = \max_{r \leq t} w_r$.[39] The Malliavin semi-elasticity is $D_r \log \theta_t = \mathbb{I}_{\{r \leq \gamma_t\}}$: the sensitivity to the shock at time r equals one prior to the time when the maximum is attained, denoted by γ_t, and zero thereafter. The optimal labor wedge is:

$$\frac{\tau_t^l}{1-\tau_t^l} = \varepsilon_t \int_0^{\gamma_t} \frac{E_r[D_r m(c_t)]}{m(c_t)} dr \tag{31}$$

and accumulates the informational advantage only to time of the maximum γ_t. The shorter horizon results in a lower labor wedge as the agent has no additional informational advantage over the planner after time γ_t.

[38] Appendix A.12 contains detailed derivations and also presents results for the Cox-Ingersoll-Ross process (Cox, Ingersoll, and Ross, 1985) which features level-dependent volatility that results in lower taxes at low skill levels and higher taxes at high skill levels. We further derive results for the general diffusion process.

[39] The notion that productivity is given by the maximum of an individual's original productivity and the best of their new ideas is akin to Jovanovic and Rob (1989), Kortum (1997), Lucas (2009), Perla and Tonetti (2014), and Lucas and Moll (2014).

8 Conclusion

Modern macroeconomics is stochastic and is built on recursive methods. While the actions in these models are history dependent, the past is represented by a small number of finite-dimensional state variables. This paper provides an alternative that allows to analyze problems with general path-dependency and opens venues to develop new models where path-dependency plays a central role. The main tool that we develop – the total derivative formula – allows to analytically characterize the dynamics of the central object in these models, the evolution of conditional expectations. We develop our results in a habit economy that is important in its own account and also serves as a prototype for general path-dependence in a variety of environments. Our methodology delivers closed-form solutions for several examples, and facilitates qualitative and quantitative comparison with benchmark models. We showcase how our methodology can yield new results in a number of important environments: robustness, limited commitment, and dynamic private information.

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Policy with Stochastic Hysteresis

Online Appendix

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A Appendices

A.1 Habit Economies

We show the generality of habit economies. We connect other settings to the feasibility constraint (1), a general objective:

$$E\int_0^T \big(u(c_t) + \varphi(x_t, h_t(x^t, w^t)) - v(n_t)\big)dt,$$

and the linearly seperable habit specification (3).

Climate. We first connect our benchmark habit model to the macro-climate model of Golosov, Hassler, Krusell, and Tsyvinski (2014). We consider an economy with a single sector, and abstract from capital accumulation to simplify the exposition. In Golosov, Hassler, Krusell, and Tsyvinski (2014) preferences for consumption are logarithmic, and the production technology is $F_t(h_t, e_t) = (1 - d_t(h_t))\tilde{F}_t(e_t)$ where h_t is the stock of emissions, e_t capture current emissions, and $1 - d_t(h_t) = \exp(-\gamma_t h_t)$ is the damage function. In this case, the planner's problem is to maximize:

$$E\int_0^T \big(\log \tilde{F}_t(e_t) - \gamma_t h_t\big)dt = E\int_0^T \big(u_t(e_t) - \gamma_t h_t\big)dt,$$

where $u_t(e_t) := \log \tilde{F}_t(e_t)$.

Our generalization allows the stock of emissions to be a general function of past emissions and past shocks. In other words, we allow the stock of emissions h_t to follow the habit specification (3), that is, $h_t(e^t, w^t) = \int_0^t k_t^s(w^t)e_s ds$. We thus allow deprecation to not only follow a deterministic pattern, but also to vary generally with depreciation shocks. Golosov, Hassler, Krusell, and Tsyvinski (2014) study a special case where the stock of emissions depends deterministically on how long ago emissions were made.

Firm Investment. To map our setting to firm investment decisions, let $\varphi(x_t, h_t(x^t, w^t))$ be the firm period profit which depends on current output produced using the stock of capital $F(h_t)$, and current investments x_t, or:

$$\varphi(x_t, h_t(x^t, w^t)) = F(h_t) - x_t$$

where productive capital h_t follows (3). As a result, the firm's profit maximization problem can be written as maximizing:

$$E \int_0^T \big(F(h_t) - x_t(w^t)\big)\, dt$$

Our generalization allows productive capital to be a general function of past investments. That is, we allow deprecation to not only follow an arbitrary deterministic depreciation pattern, but also vary generally with depreciation shocks. For example, we incorporate depreciation shocks that impact older machines but not recent vintages. The case studied by Rogerson (2008) is a special case where the production value of past investments is given by $k_t^s(w^t) = k_{t-s}$, that is, the extent of depreciation depends deterministically on how long ago the investment was made.

Knowledge Capital. We next map our environment to canonical models of knowledge capital accumulation. We distinguish two different classes of models.

First, labor productivity can be endogenized through past financial investment choices (Becker, 1962; Ben-Porath, 1967; Heckman, 1976). Let $\varphi(y_t, h_t)$ denote the utility cost to generate output y_t given knowledge capital h_t:

$$\varphi(y_t, h_t) = -v\left(y_t/h_t\right) \tag{A.1}$$

In this case, the agent's problem can be written as maximizing:

$$E \int_0^T (u(c_t) - v\left(y_t/h_t\right))dt \tag{A.2}$$

subject to the budget constraint (1). Under our generalization knowledge is a general function of past choices and experiences. For example, habit (3) allows the return on investment in education x_s to depend on an individual's current age t, their age at the time of training s, and their personal experiences.

Labor productivity can also be endogenized through past time investments, that is, learning-by-doing (Arrow, 1962; Uzawa, 1965; Lucas, 1988). Let $\varphi(y_t, h_t)$ again be the utility cost to generate effective income y_t given knowledge capital h_t in equation (A.1). However, human capital is now given by:

$$h_t(y^t, w^t) = \int_0^t k_t^s(w^t) y_s ds$$

That is, human capital is now a function of prior output, and the agent's problem is to maximize (A.2) subject to the constraint $E \int_0^T (c_t - y_t)dt \leq 0$. Under our generalization knowledge capital is a general function of previous time investments and experiences.

A.2 Malliavin Derivatives

This appendix contains a number of examples of Malliavin derivatives.

Stochastic Integral of Deterministic Function $F(w^t) = \int_0^t f_r dw_r$. Function f_r is deterministic, it does not depend on the realizations of w. Consider the variation as in (6):

$$\begin{aligned} F\Big(w^t + \varepsilon \int_0^{\cdot} z_r dr\Big) - F(w^t) &= \int_0^t f_r d\Big(w_r + \varepsilon \int_0^r z_s ds\Big) - \int_0^t f_r dw_r \\ &= \int_0^t f_r dw_r + \varepsilon \int_0^t f_r d\Big(\int_0^r z_s ds\Big) - \int_0^t f_r dw_r = \varepsilon \int_0^t f_r z_r dr \end{aligned}$$

Hence, this implies by (6) that the Malliavin derivative $D_r F(w^t) = D_r \int_0^t f_r dw_r$ is equal to f_r for $r \leq t$ and zero otherwise.

Random Riemannian Integral $F(w^t) = \int_0^t x_s(w^s) ds$. To find the Malliavin derivative of $F(w^t)$, we use the Malliavin derivative of $x_s(w^s)$. By the definition of the Malliavin derivative (6):

$$x_s\Big(w^s + \varepsilon \int_0^{\cdot} z_r dr\Big) - x_s(w^s) = \varepsilon \int_0^s D_r x_s z_r dr + o(\varepsilon),$$

As a result, we can write the variation, up to the first order, as:

$$\int_0^t x_s\Big(w^s + \varepsilon \int_0^{\cdot} z_r dr\Big) ds - \int_0^t x_s(w^s) ds = \varepsilon \int_0^t \int_0^s D_r x_s z_r dr ds = \varepsilon \int_0^t \int_r^t D_r x_s ds z_r dr$$

where the first equality follows from the Malliavin derivative of $x_s(w^s)$, and the second equality by changing the order of integration. So,

$$D_r F(w^t) = \int_r^t D_r x_s ds$$

for $r \leq t$, and zero otherwise. We can alternatively see this by formally interchanging integration and Malliavin differentiation:

$$D_r F(w^t) = D_r \int_0^t x_s ds = \int_0^t D_r x_s ds = \int_r^t D_r x_s ds$$

where the final equality uses that $D_r x_s = 0$ for $s < r$.

Ito Integral $F(w^t) = \int_0^t x_s(w^s)dw_s$. Mechanically, we consider the perturbation:

$$\int_0^t x_s\Big(w^s + \varepsilon \int_0^{\cdot} z_r dr\Big) d\Big(w_s + \varepsilon \int_0^s z_k dk\Big) - \int_0^t x_s(w^s)dw_s$$

Note the difference between the bounds on the variations. Next, we expand this expression up to first-order as:

$$\begin{aligned}
&\int_0^t x_s\Big(w^s + \varepsilon \int_0^{\cdot} z_r dr\Big) dw_s + \varepsilon \int_0^t x_s\Big(w^s + \varepsilon \int_0^{\cdot} z_r dr\Big) d\Big(\int_0^s z_k dk\Big) - \int_0^t x_s\left(w^s\right) dw_s \\
&= \varepsilon \int_0^t \Big(\int_0^s D_r x_s z_r dr\Big) dw_s + \varepsilon \int_0^t x_s\Big(w^s + \varepsilon \int_0^{\cdot} z_r dr\Big) z_s ds \\
&= \varepsilon \int_0^t \int_0^s D_r x_s z_r dr dw_s + \varepsilon \int_0^t \Big(x_s + \varepsilon \int_0^s D_r x_s z_r dr\Big) z_s ds,
\end{aligned}$$

where the second and third equality both follow from the definition of the Malliavin derivative (6). We observe that we can ignore the terms of order ε^2 to write the perturbation as:

$$\begin{aligned}
&\varepsilon \int_0^t \int_0^s D_r x_s z_r dr dw_s + \varepsilon \int_0^t x_s z_s ds = \varepsilon \int_0^t \int_r^t D_r x_s z_r dw_s dr + \varepsilon \int_0^t x_s z_s ds \\
&= \varepsilon \int_0^t \left(\int_r^t D_r x_s dw_s + x_r\right) z_r dr
\end{aligned}$$

where the first equality follows by changing the order of integration, while the second equality follows from isolating z_r and regrouping terms. So,

$$D_r F(w^t) = \int_r^t D_r x_s dw_s + x_r(w^r)$$

as long as $r \leq t$, and zero otherwise. We can alternatively see this by formally interchanging integration and Malliavin differentiation, and using the product rule:

$$D_r F(w^t) = D_r \int_0^t x_s dw_s = D_r \int_r^t x_s dw_s = \int_r^t D_r x_s dw_s + x_r(w^r).$$

A.3 Proof of Total Derivative Formula

We provide a heuristic proof for the total derivative formula as well as a rigorous proof for a generalization of the total derivative formula.

A.3.1 Heuristic Proof of Total Derivative Formula

We provide a heuristic proof for the total derivative formula in Proposition 1. The key component for the heuristic derivation is given by the Clark-Ocone formula. Applied to the random variable ξ_t, the Clark-Ocone formula gives

$$\xi_t = E[\xi_t] + \int_0^T E_r[D_r \xi_t] dw_r.$$

Taking the conditional expectation over this formula leads to:

$$\Lambda_t = E[\xi_t] + \int_0^t E_r[D_r \xi_t] dw_r + E_t \int_t^T E_r[D_r \xi_t] dw_r = E[\xi_t] + \int_0^t E_r[D_r \xi_t] dw_r.$$

Since $\xi_t = \int_0^t \partial_s \xi_s ds$, we obtain:

$$d\Lambda_t = E_t[D_t \xi_t] dw_t + \Big(E[\partial_t \xi_t] + \int_0^t E_r[D_r \partial_t \xi_t] dw_r \Big) dt. \tag{A.3}$$

To obtain the total derivative formula, we rewrite the drift term in equation (A.3) by applying the Clark-Ocone formula to the derivative $\partial_t \xi_t$:

$$\partial_t \xi_t = E[\partial_t \xi_t] + \int_0^T E_r[D_r \partial_t \xi_t] dw_r.$$

By again taking the conditional expectations:

$$E_t[\partial_t \xi_t] = E[\partial_t \xi_t] + \int_0^t E_r[D_r \partial_t \xi_t] dw_r.$$

we can write the total derivative

$$d\Lambda_t = E_t[\partial_t \xi_t] dt + E_t[D_t \xi_t] dw_t. \tag{7}$$

as in Proposition 1.

A.3.2 General Case of Total Derivative Formula

We provide a general case for the total derivative formula.

We first assume $\eta = (\eta_t)_{t \in [0,T]}$ is a measurable stochastic process that is square integrable so that $E \int_0^T \eta_t^2 dt < \infty$. Moreover, for almost all times t we assume $E\eta_t^2 < \infty$. Since the process η is square integrable it has a Clark representation, meaning there exists a progressively measurable (in s) process h_s^t, such that for all $t \in [0, T]$:

$$\eta_t = E\eta_t + \int_0^T h_s^t(w^s) dw_s$$

almost surely, where h_s^t is $\mathcal{F}_s$-measurable.

Define $\xi_t := \int_0^t \eta_s ds$. Since the process η_t is square integrable, it follows that the process ξ_t is also square integrable, and hence it has a Clark representation such that for all $t \in [0, T]$:

$$\xi_t = E\xi_t + \int_0^T g_s^t(w^s) dw_s.$$

Using these specifications, we formulate the generalized version of the total derivative formula.

Proposition 2. *Generalized Total Derivative Formula.* Let Λ_t be a process so that $\Lambda_t = E_t[\xi_t]$, where $\xi_t := \int_0^t \eta_s ds$ and the process η is square integrable. Then,

$$d\Lambda_t = E_t[\partial_t \xi_t]dt + g_t^t dw_t. \tag{A.4}$$

Proof. Consider the process for $t \in [0, T]$

$$m_t = \Lambda_t - \int_0^t E_r \eta_r dr$$

which in differential form gives that:

$$d\Lambda_t = dm_t + E_t \eta_t dt = dm_t + E_t[\partial_t \xi_t]dt \tag{A.5}$$

where the second equality follows from the observation that $\partial_t \xi_t = \eta_t$.

We next verify that the process m_t is a martingale. Indeed, for $s < t$:

$$\begin{aligned} E_s m_t =& E_s\Big[\Lambda_t - \int_0^t E_r \eta_r dr\Big] = E_s\left[E_t \xi_t\right] - \int_0^t E_s\left[E_r \eta_r\right] dr = E_s \xi_t - \int_0^s E_r \eta_r dr - \int_s^t E_s \eta_r dr \\ =& E_s\left[\xi_t - \int_s^t \eta_r dr\right] - \int_0^s E_r \eta_r dr = E_s \xi_s - \int_0^s E_r \eta_r dr = \Lambda_s - \int_0^s E_r \eta_r dr = m_s \end{aligned}$$

where we use the law of iterated expectations to obtain the third equality.

As a Brownian martingale, process m_t has a stochastic integral representation, $m_t = \int_0^t v_r dw_r$ for some progressively measurable and square integrable process v_s.[40] To find this process we use Ito isometry. For arbitrary progressively measurable square integrable process $u = (u_s)_{s \in [0,T]}$, we have

$$\begin{aligned} E\int_0^T v_s u_s ds &= E\Big[\int_0^T v_s dw_s \int_0^T u_s dw_s\Big] = E\Big[m_T \int_0^T u_s dw_s\Big] = E\Big[\Big(\xi_T - \int_0^T E_r \eta_r dr\Big) \int_0^T u_s dw_s\Big] \\ &= E\Big[\int_0^T \big(\eta_r - E_r \eta_r\big) dr \int_0^T u_s dw_s\Big] = \int_0^T E\Big[\big(\eta_r - E_r \eta_r\big) \int_0^T u_s dw_s\Big] dr \end{aligned}$$

[40]The constant $m_0 = 0$ since $m_0 = \Lambda_0 = E\xi_0 = 0$.

To develop the final term, we observe that

$$\eta_r - E_r\eta_r = E\eta_r + \int_0^T h_s^r dw_s - E\eta_r - \int_0^r h_s^r dw_s = \int_r^T h_s^r dw_s \tag{A.6}$$

Hence, by Ito isometry, we have $E\big[\,(\eta_r - E_r\eta_r)\int_0^T u_s dw_s\big] = E\int_r^T h_s^r u_s ds$. Finally,

$$E\int_0^T v_s u_s ds = E\int_0^T\int_r^T u_s h_s^r ds dr = E\int_0^T u_s \int_0^s h_s^r dr ds = E\int_0^T u_s g_s^s ds$$

where the first equality follows by substituting (A.6), the second equality follows by changing the order of integration, and where $g_s^t = \int_0^t h_s^r dr$ is a continuous modification for the process g_s^t. As a result, $v_s = g_s^s$, or $v_t = g_t^t$. Since $m_t = \int_0^t v_r dw_r$, we have that $dm_t = v_t dw_t = g_t^t dw_t$. Substituting into (A.5), we obtain equation (A.4). □

A.4 Total Derivative Formula

This appendix presents two special cases of the total derivative formula, and further shows that the assumption of time differentiability on the process ξ_t can be significantly relaxed.

A.4.1 Time-Invariant ξ

We first consider the case where $\xi_t = \xi$ is Malliavin differentiable but does not vary with time. In this case, the Clark-Ocone formula is a nested case of our total derivative formula (7). Using the total derivative formula (7), we see that when $\xi_t = \xi$ the time derivative is equal to zero so that we obtain:

$$dX_t = E_t[D_t\xi]dw_t.$$

Alternatively, we can use the Clark-Ocone representation for the random variable ξ to write:

$$\xi = E\xi + \int_0^T E_r[D_r\xi]dw_r$$

and hence, we obtain that

$$X_t = E_t\xi = E\xi + \int_0^t E_r[D_r\xi]dw_r$$

where the second equality follows by the law of iterated expectations, and because the expectation over future innovations equals zero. Hence, the Clark-Ocone representation gives the identical dynamics of optimal policy, $dX_t = E_t[D_t\xi]dw_t$, when ξ_t is time-invariant.

A.4.2 Time-Invariant Habit Coefficients

We next characterize the drift and diffusion decomposition of optimal policy with time-invariant habit coefficients, by which we mean that the habit coefficient does not depend on t, or $k_s^t(w^s) = k_s(w^s)$. Using the Clark-Ocone formula, we first write optimal policy (5) as:

$$\Lambda_t = E_t \int_t^T k_r(w^r)dr = E_t \int_0^T k_r(w^r)dr - \int_0^t k_r(w^r)ds. \tag{A.7}$$

We next observe $m_t = E_t \int_0^T k_r(w^r)dr$ is a martingale since

$$E_t m_s = E_t\Big[E_s \int_0^T k_r(w^r)dr\Big] = E_t \int_0^T k_r(w^r)dr = m_t \tag{A.8}$$

for $s > t$, where the second equality follows from the law of iterated expectations.

By the Clark-Ocone formula we obtain an explicit formulation of the random variable m_T:

$$m_T = Em_T + \int_0^T E_r[D_r m_T]dw_r. \tag{A.9}$$

so that $m_t = E_t m_T = Em_T + \int_0^t E_r[D_r m_T]dw_r$. As a result, we can write:

$$m_t = E_t m_T = m_0 + \int_0^t E_r\Big[D_r \int_0^T k_s(w^s)ds\Big]dw_r = m_0 + \int_0^t E_r\Big[\int_r^T D_r k_s(w^s)ds\Big]dw_r, \tag{A.10}$$

where the final equality uses $D_r \int_0^T k_s(w^s)ds = \int_r^T D_r k_s(w^s)ds$. Intuitively, a perturbation of the stochastic trajectory at time r does not affect habit coefficients at earlier times $s < r$ and hence the Malliavin derivative for those times is zero.

Following equations (A.7) and (A.8) we write that $d\Lambda_t = dm_t - k_t(w^t)dt$. Moreover, using (A.10) we conclude:

$$d\Lambda_t = -k_t(w^t)dt + E_t\Big[\int_t^T D_t k_s(w^s)ds\Big]dw_t.$$

If we follow the drift and diffusion decomposition of optimal policy (8) immediately, we obtain $\partial_t k_s^t(w^s) = 0$ with time-invariant habit coefficients, so the drift coefficient is $-k_t(w^t)$. In sum, we establish the same result.

A.4.3 Time Differentiability

In this appendix we show that our mild assumption that habit coefficients are differentiable with respect to time can be significantly relaxed. The idea is that the when ξ_t is not differentiable with respect to time it can be replaced by another process $\hat{\xi}_t$ where $\hat{\xi}_t$ is differentiable with respect to time such that $E_t\xi_t = E_t\hat{\xi}_t$. We illustrate this powerful idea by means of two examples.

Example 1: $\xi_t = w_t$. First, consider the case where the process ξ_t is given by the Brownian motion. In this case, we consider the process $\hat{\xi}_t = w_T$, the terminal value of the Brownian process. Since the Brownian motion is a martingale $w_t = E_t w_t = E_t w_T$ implying $E_t\xi_t = E_t\hat{\xi}_t$. By applying the total derivative formula (7) with $\hat{\xi}_t$, we obtain

$$dE_t\xi_t = E_t[\partial_t\hat{\xi}_t]dt + E_t[D_t\hat{\xi}_t]dw_t = E_t[D_t w_T]dw_t = dw_t$$

where $D_t w_T = 1$ intuitively means that the perturbation of the path at time t changes the terminal value w_T by 1.

Example 2: $k_s^t(w^s) = w_t$. As a second example, we consider the case where the habit coefficients in optimal policy (5) are given by the realization of the Brownian motion $k_s^t(w^s) = w_t$ so that the habit coefficients are not differentiable with respect to time $\xi_t = \int_t^T k_s^t(w^s)ds = \int_t^T w_t ds$. In this case, we consider the process $\hat{\xi}_t = (T-t)w_T$ so that:

$$E_t\xi_t = E_t\int_t^T w_t ds = \int_t^T w_t ds = (T-t)w_t = E_t\big[(T-t)w_T\big] = E_t\hat{\xi}_t.$$

where the fourth equality follow since the Brownian motion is a martingale. By applying the total derivative formula (7) with $\hat{\xi}_t$, we obtain:

$$dE_t\xi_t = E_t[\partial_t\hat{\xi}_t]dt + E_t[D_t\hat{\xi}_t]dw_t = -E_t[w_T]dt + (T-t)dw_t = -w_t dt + (T-t)dw_t,$$

where we use $D_t w_T = 1$. Therefore, even though the habit coefficients are not differentiable with respect to time, the total derivative formula can be used to characterize the evolution of optimal policy.

A.5 Unfamiliarity

We describe the formal analysis for the stochastic habits under unfamiliarity.

A.5.1 Ornstein-Uhlenbeck Process

We start by discussing features of the Ornstein-Uhlenbeck process that we use to derive our results. Let x_t be a stationary Ornstein-Uhlenbeck process, defined by the stochastic differential equation

$dx_t = \lambda(\mu - x_t)dt + \sigma dw_t$, where w_t is the Brownian motion. The stochastic differential equation for the process x_t can be solved to obtain:

$$x_t = x_0 e^{-\lambda t} + \mu(1 - e^{-\lambda t}) + \sigma \int_0^t e^{-\lambda(t-r)} dw_r.$$

We first note that the Malliavin derivative is given by, for $t < s$:

$$D_t x_s = \sigma e^{-\lambda(s-t)} \tag{A.11}$$

Second, starting with time t, we can similarly write:

$$x_s = x_t e^{-\lambda(s-t)} + \mu\big(1 - e^{-\lambda(s-t)}\big) + \sigma \int_0^{s-t} e^{-\lambda(s-t-r)} d\tilde{w}_r,$$

where $\tilde{w}_r = w_{r+t} - w_t$, and observe that conditional on time t, x_s is a normally distributed random variable.

For our analysis, it is key to understand the distribution of the random variable $x_s - \frac{1}{t}\int_0^t x_r dr$ for $s > t$ conditional on information at time t. The conditional mean of the random variable x_s at time t is given by $E_t x_s = x_t e^{-\lambda(s-t)} + \mu(1 - e^{-\lambda(s-t)})$. In turn, this implies that the conditional mean of $x_s - \frac{1}{t}\int_0^t x_r dr$, which we denote by $\hat{m}_{st}$, equals:

$$\hat{m}_{st} = E_t\left[x_s - \frac{1}{t}\int_0^t x_r dr\right] = x_t e^{-\lambda(s-t)} + \mu\big(1 - e^{-\lambda(s-t)}\big) - \frac{1}{t}\int_0^t x_r dr. \tag{A.12}$$

and, furthermore, that

$$x_s - E_t x_s = \sigma \int_0^{s-t} e^{-\lambda(s-t-r)} d\tilde{w}_r.$$

We next compute the conditional variance of the variable $x_s - \frac{1}{t}\int_0^t x_r dr$, which we denote by $\hat{\sigma}^2_{st}$:

$$\begin{aligned} \hat{\sigma}^2_{st} &= E_t\left[\Big(x_s - E_t x_s\Big)^2\right] = \sigma^2 E_t\left[\Big(\int_0^{s-t} e^{-\lambda(s-t-r)} d\tilde{w}_r\Big)^2\right] = \sigma^2 E_t\left[\int_0^{s-t} e^{-2\lambda(s-t-r)} dr\right] \\ &= \frac{\sigma^2}{2\lambda}\Big(1 - e^{-2\lambda(s-t)}\Big) \end{aligned} \tag{A.13}$$

where the third equality follows from Ito isometry. Hence, $x_s - \frac{1}{t}\int_0^t x_r dr$ is a normally distributed random variable with the mean given by $\hat{m}_{st}$ in (A.12), and the conditional variance $\hat{\sigma}^2_{st}$ given in (A.13). Let $z \sim N(0,1)$ be a standard normal variable, then conditional on information at time t, we equivalently write $x_s - \frac{1}{t}\int_0^t x_r dr \sim \hat{m}_{st} + \hat{\sigma}_{st} z$, where z is independent from the filtration generated by the process w up to time t, which we denote by $\mathcal{F}_t$. Going forward, we drop the subscript st on the mean and the conditional variance when this is with limited risk of confusion.

A.5.2 Optimal Policy

To characterize optimal policy under unfamiliarity, we recall that the habit coefficients in the case of unfamiliarity (9) are given by $k_t^s = \alpha \exp(-\delta(t-s) - \kappa \big| \log \theta_t - \log \bar{\theta}_s \big| (t-s))$, and hence the corresponding quantity:

$$\Lambda_t = E_t \int_t^T k_s^t ds = \alpha E_t \int_t^T \exp(-\delta(s-t) - \kappa | \log \theta_s - \log \bar{\theta}_t | (s-t)) ds.$$

Since logarithmic productivity follows an Ornstein-Uhlenbeck process, $\log \theta_s - \log \bar{\theta}_t$ conditional on time t is a normally distributed random variable such that $\log \theta_s - \log \bar{\theta}_t \sim \hat{m} + \hat{\sigma} z$. We can first rewrite the habit consumption wedge as:

$$\Lambda_t = \alpha \int_t^T \exp(-\hat{\delta}) E_t \exp\big(- \hat{\kappa} \big| \log \theta_s - \log \bar{\theta}_t \big| \big) ds,$$

where $\hat{\delta} := \delta(s-t)$ and $\hat{\kappa} := \kappa(s-t)$. To characterize optimal policy, we focus our attention to characterizing $\chi_t := E_t \exp(-\hat{\kappa} | \log \theta_s - \log \bar{\theta}_t |)$:

$$\chi_t = E_t \exp\big(- \hat{\kappa} \big| \log \theta_s - \log \bar{\theta}_t \big| \big) = E_t \exp\big(- \hat{\kappa} \big| \hat{m} + \hat{\sigma} z \big| \big). \tag{A.14}$$

This expression is simplified by evaluating the expectation:

$$\begin{aligned} \chi_t &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\hat{\kappa}|\hat{m}+\hat{\sigma} z| - \frac{1}{2} z^2} dz = \frac{1}{\sqrt{2\pi\hat{\sigma}^2}} \int_{-\infty}^{\infty} e^{-\hat{\kappa}|\hat{z}| - \frac{1}{2}\left(\frac{\hat{z}-\hat{m}}{\hat{\sigma}}\right)^2} d\hat{z} \\ &= \frac{1}{\sqrt{2\pi\hat{\sigma}^2}} \int_0^{\infty} e^{-\hat{\kappa}\hat{z} - \frac{1}{2}\left(\frac{\hat{z}-\hat{m}}{\hat{\sigma}}\right)^2} d\hat{z} + \frac{1}{\sqrt{2\pi\hat{\sigma}^2}} \int_{-\infty}^{0} e^{\hat{\kappa}\hat{z} - \frac{1}{2}\left(\frac{\hat{z}-\hat{m}}{\hat{\sigma}}\right)^2} d\hat{z} \\ &= \frac{1}{\sqrt{2\pi\hat{\sigma}^2}} \int_0^{\infty} e^{-\hat{\kappa}\hat{z} - \frac{1}{2}\left(\frac{\hat{z}-\hat{m}}{\hat{\sigma}}\right)^2} d\hat{z} + \frac{1}{\sqrt{2\pi\hat{\sigma}^2}} \int_0^{\infty} e^{-\hat{\kappa}\hat{z} - \frac{1}{2}\left(\frac{\hat{z}+\hat{m}}{\hat{\sigma}}\right)^2} d\hat{z} = \chi_t^+ + \chi_t^-, \end{aligned} \tag{A.15}$$

where the second equality follows by a change of variable $\hat{z} := \hat{m} + \hat{\sigma} z$. The third equality follows by splitting the expectation conditional on $\hat{z}$ being positive (the first term), and on $\hat{z}$ being negative (the second term). We next simplify separately the conditional expectation conditional on $\hat{z}$ being positive (χ_t^+), and on $\hat{z}$ being negative (χ_t^-).

First, we simplify the expectation conditional on $\hat{z}$ being positive:

$$\begin{aligned} \chi_t^+ &= \frac{1}{\sqrt{2\pi\hat{\sigma}^2}} \int_0^{\infty} e^{-\hat{\kappa}\hat{z} - \frac{1}{2}\left(\frac{\hat{z}-\hat{m}}{\hat{\sigma}}\right)^2} d\hat{z} = \frac{1}{\sqrt{2\pi\hat{\sigma}^2}} \int_0^{\infty} e^{-\frac{2\hat{\kappa}\hat{z}\hat{\sigma}^2 + \hat{z}^2 - 2\hat{z}\hat{m} + \hat{m}^2}{2\hat{\sigma}^2}} d\hat{z} \\ &= \frac{1}{\sqrt{2\pi\hat{\sigma}^2}} \int_0^{\infty} e^{-\frac{(\hat{z}+\hat{\kappa}\hat{\sigma}^2-\hat{m})^2 - (\hat{\kappa}\hat{\sigma}^2)^2 + 2\hat{m}\hat{\kappa}\hat{\sigma}^2}{2\hat{\sigma}^2}} d\hat{z} = \frac{e^{\frac{\hat{\kappa}^2\hat{\sigma}^2}{2} - \hat{m}\hat{\kappa}}}{\sqrt{2\pi\hat{\sigma}^2}} \int_0^{\infty} e^{-\frac{(\hat{z}+\hat{\kappa}\hat{\sigma}^2-\hat{m})^2}{2\hat{\sigma}^2}} d\hat{z} \end{aligned}$$

To simplify the expression further, we apply a change of variables $\tilde{z} = -\frac{\hat{z}+\hat{\kappa}\hat{\sigma}^2-\hat{m}}{\hat{\sigma}}$ to write:

$$\chi_t^+ = \frac{e^{\frac{\hat{\kappa}^2\hat{\sigma}^2}{2}-\hat{m}\hat{\kappa}}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma}} e^{-\frac{1}{2}\tilde{z}^2} d\tilde{z} = \exp\left(\frac{\hat{\kappa}^2\hat{\sigma}^2}{2} - \hat{m}\hat{\kappa}\right)\Phi\left(\frac{\hat{m}}{\hat{\sigma}} - \hat{\kappa}\hat{\sigma}\right),$$

where Φ denotes the cumulative distribution function for the standard normal distribution, that is, $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{\tilde{z}^2}{2}} d\tilde{z}$.

Second, we simplify the expectation conditional on $\hat{z}$ being negative, the second term in (A.15). We observe that the second term in (A.15) is equivalent to the first term in (A.15) up to changing $\hat{m}$ to $-\hat{m}$, which immediately allows us to write:

$$\chi_t^- = \exp\left(\frac{\hat{\kappa}^2\hat{\sigma}^2}{2} + \hat{m}\hat{\kappa}\right)\Phi\left(-\frac{\hat{m}}{\hat{\sigma}} - \hat{\kappa}\hat{\sigma}\right).$$

As a result, we can finally write χ_t of (A.14) as:

$$\chi_t = \exp\left(\frac{\hat{\kappa}^2\hat{\sigma}^2}{2}\right)\left(\exp(-\hat{m}\hat{\kappa})\Phi\left(\frac{\hat{m}}{\hat{\sigma}} - \hat{\kappa}\hat{\sigma}\right) + \exp(\hat{m}\hat{\kappa})\Phi\left(-\frac{\hat{m}}{\hat{\sigma}} - \hat{\kappa}\hat{\sigma}\right)\right), \tag{A.16}$$

which thus fully characterizes the habit wedge Λ_t.

A.5.3 Drift and Diffusion Decomposition

We now analyze the evolution for the habit wedge using (8). In turn, we identify the drift coefficient and the diffusion coefficient.

Drift. The drift coefficient in the drift and diffusion decomposition (8) is $E_t \int_t^T \partial_t k_s^t ds - k_t^t$, where we recall $k_s^t = \alpha\exp(-\hat{\delta} - \hat{\kappa}|\log\theta_s - \frac{1}{t}\int_0^t \log\theta_r dr|)$, so:

$$\partial_t k_s^t = k_s^t\big(\delta + \kappa\left|\log\theta_s - \log\bar{\theta}_t\right| + \hat{\kappa}\frac{a_t}{t}\text{sign}\left(\log\theta_s - \log\bar{\theta}_t\right)\big).$$

where $a_t := \log\theta_t - \log\bar{\theta}_t$.

Since logarithmic productivity follows an Ornstein-Uhlenbeck process, conditional on time t it follows that $\log\theta_s - \log\bar{\theta}_t \sim \hat{m} + \hat{\sigma}z$, and we write:

$$E_t\partial_t k_s^t = \delta E_t k_s^t + \alpha\kappa e^{-\hat{\delta}}E_t\left[|\hat{m}+\hat{\sigma}z|\, e^{-\hat{\kappa}|\hat{m}+\hat{\sigma}z|}\right] + \alpha\hat{\kappa}\frac{a_t}{t}e^{-\hat{\delta}}E_t\left[\text{sign}(\hat{m}+\hat{\sigma}z)e^{-\hat{\kappa}|\hat{m}+\hat{\sigma}z|}\right]. \tag{A.17}$$

We next simplify the drift coefficient by analyzing the second and third term, since we fully characterized the first term in the previous subsection. To do so, we start with χ_t in (A.14) and its characterization in (A.16). We differentiate both sides of (A.16) with respect to $\hat{m}$. First, we differentiate the left-hand side to obtain:

$$-\hat{\kappa}E_t\left[e^{-\hat{\kappa}|\hat{m}+\hat{\sigma}z|}\text{sign}(\hat{m}+\hat{\sigma}z)\right].$$

Second, we differentiate the right-hand side to obtain:

$$e^{\frac{\hat{\kappa}^2\hat{\sigma}^2}{2}}\left[e^{-\hat{m}\hat{\kappa}}\Big(-\hat{\kappa}\Phi\Big(\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma}\Big)+\frac{1}{\hat{\sigma}}\varphi\Big(\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma}\Big)\Big)+e^{\hat{m}\hat{\kappa}}\Big(\hat{\kappa}\Phi\Big(-\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma}\Big)-\frac{1}{\hat{\sigma}}\varphi\Big(-\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma}\Big)\Big)\right]$$

where φ is the probability density function for a standard normal distribution, or $\varphi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. We observe that the terms regarding the probability density function cancel as:

$$e^{-\frac{\left(\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma}\right)^2+2\hat{\kappa}\hat{m}}{2}}-e^{-\frac{\left(\frac{\hat{m}}{\hat{\sigma}}+\hat{\kappa}\hat{\sigma}\right)^2-2\hat{\kappa}\hat{m}}{2}}=e^{-\frac{\left(\frac{\hat{m}}{\hat{\sigma}}\right)^2+(\hat{\kappa}\hat{\sigma})^2}{2}}-e^{-\frac{\left(\frac{\hat{m}}{\hat{\sigma}}\right)^2+(\hat{\kappa}\hat{\sigma})^2}{2}}=0.$$

We combine the derivatives of the left and right-hand side, and divide by $-\hat{\kappa}$ to obtain:

$$E_t\left[\text{sign}(\hat{m}+\hat{\sigma}z)e^{-\hat{\kappa}|\hat{m}+\hat{\sigma}z|}\right]=e^{\frac{\hat{\kappa}^2\hat{\sigma}^2}{2}}\left(e^{-\hat{m}\hat{\kappa}}\Phi\left(\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma}\right)-e^{\hat{m}\hat{\kappa}}\Phi\left(-\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma}\right)\right)\tag{A.18}$$

which takes care of the third term in equation (A.17).

To take care of the second term in equation (A.17) we start with the characterization of χ_t in (A.16), which we now differentiate with respect to $\hat{\kappa}$. First, we differentiate the left-hand side:

$$-E_t\left[e^{-\hat{\kappa}|\hat{m}+\hat{\sigma}z|}\,|\hat{m}+\hat{\sigma}z|\right].$$

Second, we differentiate the right-hand side to obtain:

$$\begin{aligned}&\hat{\kappa}\hat{\sigma}^2e^{\frac{\hat{\kappa}^2\hat{\sigma}^2}{2}}\Big(e^{-\hat{m}\hat{\kappa}}\Phi\Big(\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma}\Big)+e^{\hat{m}\hat{\kappa}}\Phi\Big(-\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma}\Big)\Big)\\&+e^{\frac{\hat{\kappa}^2\hat{\sigma}^2}{2}}\Big[e^{-\hat{m}\hat{\kappa}}\Big(-\hat{m}\Phi\Big(\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma}\Big)-\hat{\sigma}\varphi\Big(\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma}\Big)\Big)+e^{\hat{m}\hat{\kappa}}\Big(\hat{m}\Phi\Big(-\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma}\Big)-\hat{\sigma}\varphi\Big(-\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma}\Big)\Big)\Big]\end{aligned}$$

To simplify, we use the probability density function for a standard normal distribution $\varphi(x)$ to write:

$$e^{\frac{\hat{\kappa}^2\hat{\sigma}^2}{2}}e^{-\hat{m}\hat{\kappa}}\frac{1}{\sqrt{2\pi}}e^{-\frac{(\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma})^2}{2}}=\frac{1}{\sqrt{2\pi}}e^{-\frac{(\frac{\hat{m}}{\hat{\sigma}})^2}{2}}=\varphi\Big(\frac{\hat{m}}{\hat{\sigma}}\Big),$$

and similarly that:

$$e^{\frac{\hat{\kappa}^2\hat{\sigma}^2}{2}}e^{\hat{m}\hat{\kappa}}\frac{1}{\sqrt{2\pi}}e^{-\frac{(-\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma})^2}{2}}=\frac{1}{\sqrt{2\pi}}e^{-\frac{(\frac{\hat{m}}{\hat{\sigma}})^2}{2}}=\varphi\Big(\frac{\hat{m}}{\hat{\sigma}}\Big).$$

We combine the derivative of the left-hand side and the right-hand side to write:

$$\begin{aligned}E_t\left[e^{-\hat{\kappa}|\hat{m}+\hat{\sigma}z|}\,|\hat{m}+\hat{\sigma}z|\right]=\hat{\sigma}e^{\frac{\hat{\kappa}^2\hat{\sigma}^2}{2}}&\Big(\Big(\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma}\Big)e^{-\hat{m}\hat{\kappa}}\Phi\Big(\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma}\Big)+e^{\hat{m}\hat{\kappa}}\Big(-\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma}\Big)\Phi\Big(-\frac{\hat{m}}{\hat{\sigma}}-\hat{\kappa}\hat{\sigma}\Big)\Big)\\&+2\hat{\sigma}\varphi\Big(\frac{\hat{m}}{\hat{\sigma}}\Big)\end{aligned}$$

This takes care of the second term in (A.17). As a result, the drift coefficient is characterized in terms of model primitives.

Diffusion. We derive the diffusion coefficient. Recall $k_s^t = \alpha \exp\big(-\hat{\delta} - \hat{\kappa}\big|\log\theta_s - \log\bar{\theta}_t\big|\big)$, so:

$$D_t k_s^t = k_s^t \left(-\hat{\kappa}\operatorname{sign}(\log\theta_s - \log\bar{\theta}_t)\right) D_t \log\theta_s$$

By the Malliavin derivative for the Ornstein-Uhlenbeck process (A.11), we know that $D_t \log\theta_s = \sigma e^{-\lambda(s-t)}$. Therefore, we can write:

$$E_t D_t k_s^t = -\hat{\kappa}\alpha\sigma \exp\big(-(\lambda+\delta)(s-t)\big) E_t\big[\exp\big(-\hat{\kappa}\big|\hat{m}+\hat{\sigma}z\big|\big)\operatorname{sign}(\hat{m}+\hat{\sigma}z)\big].$$

Using the characterization in (A.18), we can write this as:

$$E_t D_t k_s^t = -\hat{\kappa}\alpha\sigma \exp\big(-(\lambda+\delta)(s-t)\big) e^{\frac{\hat{\kappa}^2\hat{\sigma}^2}{2}} \Big(e^{-\hat{m}\hat{\kappa}}\Phi\Big(\frac{\hat{m}}{\hat{\sigma}} - \hat{\kappa}\hat{\sigma}\Big) - e^{\hat{m}\hat{\kappa}}\Phi\Big(-\frac{\hat{m}}{\hat{\sigma}} - \hat{\kappa}\hat{\sigma}\Big)\Big).$$

As a result, the diffusion coefficient is characterized in terms of model primitives.

A.6 Tipping Point

In this appendix we analyze stochastic habits under tipping points.

A.6.1 Ito Process

We first introduce the maximum of a Brownian motion, which we use to show that the habit wedge is an Ito process. To show that the habit wedge is an Ito process, we use the general total derivative formula (A.4). In order to apply the total derivative formula, we next establish that ξ_t is differentiable with respect to time, and that η_t is square integrable.

Let γ_t be the time when the Brownian motion achieves its maximum value on the interval $[0, t]$. Formally, we define:

$$\gamma_t = \min\{r \in [0,t] : w_r = M_t\},$$

where $M_t = \max\limits_{r\in[0,t]} w_r$. Analogously, we define $\gamma_{s,t}$ as the time when the Brownian motion attains its maximum on the interval $[s, t]$:

$$\gamma_{s,t} = \min\{r \in [s,t] : w_r = M_{s,t}\},$$

with $M_{s,t} = \max\limits_{r\in[s,t]} w_r$.

Given a twice continuously differentiable function f satisfying $f(x) = 0$ for all negative values $x \leq 0$, we consider the process:

$$\xi_t = \int_t^T f(t - \gamma_s)ds.$$

We note that $\xi_0 = 0$ since evaluated at $t = 0$ all $t - \gamma_s \leq 0$.

Our goal is to establish that the conditional expectation process $\Lambda_t = E_t\xi_t$ is an Ito process and find its drift and diffusion decomposition. The main representation we use is

$$\xi_t = \int_t^T f(t - \gamma_s)ds = \int_t^T \int_0^{\max(t-\gamma_s,0)} f'(x)dxds = \int_t^T \int_0^t f'(x)\mathbb{I}_{\{\gamma_s \leq\, t-x\}}dxds. \tag{A.19}$$

To show that $\eta_t = \partial_t\xi_t$ exists and is square integrable, we first establish that the process ξ_t is absolutely continuous. We show absolute continuity by establishing that it is Lipschitz continuous. For $s < t$ we indeed have

$$\begin{aligned}\left|\xi_t - \xi_s\right| = \left|\int_t^T f(t - \gamma_r)dr - \int_s^T f(s - \gamma_r)dr\right| &\leq \int_t^T \left|f(t - \gamma_r) - f(s - \gamma_r)\right|dr + \int_s^t \left|f(s - \gamma_r)\right|dr \\ \leq C_1 \int_t^T (t - s)dr + C_0(t - s) &\leq (TC_1 + C_0)(t - s),\end{aligned}$$

where the first equality follows by the definition and the first inequality is immediate. The second inequality is given when $C_0 = \max\limits_{0\leq r\leq T} |f(r)|$, and when $C_1 = \max\limits_{0\leq r\leq T} |f'(r)|$. The Lipschitz constant is thus given by $TC_1 + C_0$. Moreover, we observe that this implies:

$$\frac{\left|\xi_t - \xi_s\right|}{t - s} = \frac{\left|\int_0^t \eta_r dr - \int_0^s \eta_r dr\right|}{t - s} \leq (TC_1 + C_0),$$

where the equality follows since ξ_t is Lipschitz with respect to time as shown above. Taking the limits as t tends to s, we obtain $|\eta_s| \leq (TC_1 + C_0)$, from which it directly follows that η_s is square integrable. By the generalized total derivative formula (A.4) Λ_t is an Ito process. We next provide an explicit representation of this Ito process.

A.6.2 Optimal Policy

We next characterize optimal policy Λ_t. Using (A.19) we can write:

$$\Lambda_t = E_t\xi_t = \int_t^T \int_0^t f'(x)E_t\left[\mathbb{I}_{\{\gamma_s\leq t-x\}}\right] dxds = \int_t^T \int_0^t f'(x)P_t\left[\gamma_s \leq t - x\right] dxds, \tag{A.20}$$

where P_t is the conditional probability given $\mathcal{F}_t$ and $s \geq t$. We note that the time of the maximum is before time $t - x$, or $\gamma_s \leq t - x$ if and only if the maximum value on the interval $[0, t - x]$ weakly exceeds the maximum over the interval $[0,s]$, that is, $M_{0,t-x} = M_{0,s} = \max(M_{0,t}, M_{t,s})$, where the final equality follows by the definition of the maximum on an interval $[0,s]$. Hence,

$$\begin{aligned} P_t\left[\gamma_s \le t-x\right] &= P_t\left[M_{0,t-x} = \max(M_{0,t}, M_{t,s})\right] = \mathbb{I}_{\{M_{0,t-x}=M_{0,t}\}} P_t\left[M_{0,t-x} \ge M_{t,s}\right] \\ &= \mathbb{I}_{\{M_{0,t-x}=M_{0,t}\}} P_t\left[M_{0,t-x} - w_t \ge M_{t,s} - w_t\right]. \end{aligned} \tag{A.21}$$

where we observe by the definition of $M_{t,s}$ that $M_{t,s} - w_t = \max_{q\in[t,s]}(w_q - w_t) = \max_{q\in[0,s-t]}(w_{t+q} - w_t)$ is a maximum $\tilde{M}_{0,s-t}$ of a Brownian motion $\tilde{w}_q = w_{t+q} - w_t$, where $\tilde{w}$ is independent from $\mathcal{F}_t$. Since the density of the maximum of a Brownian motion $\tilde{M}_{0,s-t}$ is $\psi_{s-t}(z) = \sqrt{\frac{2}{\pi(s-t)}} e^{-\frac{z^2}{2(s-t)}}$ we write:

$$\begin{aligned} P_t\left[\gamma_s \le t-x\right] &= \mathbb{I}_{\{M_{0,t-x}\ge M_{0,t}\}} P_t\left[M_{0,t-x} - w_t \ge M_{t,s} - w_t\right] = \mathbb{I}_{\{\gamma_t \le t-x\}} \int_0^{M_{0,t-x}-w_t} \psi_{s-t}(z)dz \\ &= \mathbb{I}_{\{\gamma_t \le t-x\}} \sqrt{\frac{2}{\pi(s-t)}} \int_0^{M_{0,t-x}-w_t} e^{-\frac{z^2}{2(s-t)}} dz = 2\mathbb{I}_{\{\gamma_t \le t-x\}} \frac{1}{\sqrt{2\pi}} \int_0^{\chi} e^{-\frac{\hat{z}^2}{2}} d\hat{z} \end{aligned}$$

where the final equality follows from a change of variables such that $\hat{z} = \frac{z}{\sqrt{s-t}}$ and, hence, $d\hat{z} = \frac{dz}{\sqrt{s-t}}$, and finally $\chi := \frac{M_{0,t-x}-w_t}{\sqrt{s-t}}$. Moreover, we can write the final expression as:

$$P_t\left[\gamma_s \le t-x\right] = 2\mathbb{I}_{\{\gamma_t \le t-x\}} \frac{1}{\sqrt{2\pi}} \int_0^{\chi} e^{-\frac{\hat{z}^2}{2}} d\hat{z} = 2\mathbb{I}_{\{\gamma_t \le t-x\}} \left(\Phi(\chi) - \frac{1}{2}\right) = \mathbb{I}_{\{\gamma_t \le t-x\}} (2\Phi\left(\chi\right) - 1). \tag{A.22}$$

We can substitute the expression for the conditional probability into the expression for the habit consumption wedge (A.20) to write:

$$\Lambda_t = \int_t^T \int_0^{t-\gamma_t} f'(x)(2\Phi\left(\chi\right) - 1)dxds, \tag{A.23}$$

which provides a closed-form expression for optimal policy. We next provide the drift and diffusion decomposition.

A.6.3 Drift

By the total derivative formula, the drift in optimal policy Λ_t is given by the conditional expectation of the time derivative of ξ_t, that is, $\partial_t \xi_t$. Since $\xi_t = \int_t^T f(t-\gamma_s)ds$, the derivative with respect to time is, using Leibniz' integral rule, given by

$$\partial_t \xi_t = -f(t-\gamma_t) + \int_t^T f'(t-\gamma_s)ds.$$

Therefore, the conditional expectation of the time derivative is:

$$E_t \partial_t \xi_t = -f(t-\gamma_t) + \int_t^T E_t f'(t-\gamma_s) ds. \tag{A.24}$$

When we differentiate with respect to time, the instantaneous impact on the habit externality is eliminated, and future effects are affected to the extent that the time since the time of your life increases.

To make the conditional expectation explicit, we use a representation akin to (A.19):

$$f'(t-\gamma_s) = \int_0^{\max(t-\gamma_s,0)} f''(x)dx = \int_0^t f''(x)\mathbb{I}_{\{\gamma_s \leq\, t-x\}}dx$$

since $f'(0) = 0$. And hence it follows that:

$$E_t f'(t-\gamma_s) = \int_0^t f''(x)E_t[\mathbb{I}_{\{\gamma_s \leq\, t-x\}}]dx = \int_0^t f''(x)P_t\{\gamma_s \leq\ t-x\}dx$$

Using the characterization of the conditional probability in (A.22), we write:

$$E_t f'(t-\gamma_s) = \int_0^t f''(x)\mathbb{I}_{\{\gamma_t \leq t-x\}}\big(2\Phi(\chi)-1\big)dx = \int_0^{t-\gamma_t} f''(x)\big(2\Phi(\chi)-1\big)dx$$

By substituting back into the conditional expectation of the time derivative (A.24), we obtain:

$$E_t \partial_t \xi_t = -f(t-\gamma_t) + \int_t^T \int_0^{t-\gamma_t} f''(x)\big(2\Phi(\chi)-1\big)dxds \tag{A.25}$$

A.6.4 Diffusion

We next characterize the diffusion coefficient for the evolution of optimal policy Λ_t. We construct the Clark representation of the habit externality $\xi_t = \int_t^T \int_0^t f'(x)\mathbb{I}_{\{\gamma_s \leq\, t-x\}}dxds$. That is, we represent the value of the habit externality ξ_t as a stochastic integral. Given the formulation (A.19), this requires us to provide the Clark representation of $\mathbb{I}_{\{\gamma_s \leq t\}}$ where $t < s$.

Clark Representation of $\mathbb{I}_{\{\gamma_s \leq t\}}$. We use the observation that $\gamma_s \leq t$ if and only if $M_{0,t} \geq M_{t,s}$. Next, define the function f_n as:

$$f_n(x) = \begin{cases} 1 & x \geq 0 \\ n\big(x+\frac{1}{n}\big), & -\frac{1}{n} \leq x \leq 0 \\ 0, & x \leq -\frac{1}{n} \end{cases}$$

Hence, $\lim_{n\to\infty} f_n(x) = \mathbb{I}_{\{x\geq 0\}}$. Then, we write:

$$\mathbb{I}_{\{\gamma_s \leq t\}} = \mathbb{I}_{\{M_{0,t}\geq M_{t,s}\}} = \mathbb{I}_{\{M_{0,t}-M_{t,s}\geq 0\}} = \lim_{n\to\infty} f_n(M_{0,t} - M_{t,s}). \tag{A.26}$$

First, we find the Clark representation of $f_n(M_{0,t} - M_{t,s})$. To derive this Clark representation, we first observe that the Malliavin derivative for the maximum of a Brownian motion is known. Specifically, for $a \leq b$, $D_r M_{a,b} = \mathbb{I}_{\{r\leq\gamma_{a,b}\}}$. Intuitively, if the Brownian motion is slightly increased before the maximum is attained, the maximum value increases one-for-one. Moreover, we observe $f_n'(x) = n\mathbb{I}_{[-\frac{1}{n},0]}(x)$.

Since the maximum of the Brownian motion is Malliavin differentiable, we know by Clark-Ocone that:

$$\begin{aligned}
f_n\left(M_{0,t} - M_{t,s}\right) &= Ef_n\left(M_{0,t} - M_{t,s}\right) + \int_0^s E_r\Big[D_r f_n\left(M_{0,t} - M_{t,s}\right)\Big]dw_r \\
&= Ef_n\left(M_{0,t} - M_{t,s}\right) + \int_0^s E_r\Big[f_n'\left(M_{0,t} - M_{t,s}\right)\left(\mathbb{I}_{r\leq\gamma_{0,t}} - \mathbb{I}_{r\leq\gamma_{t,s}}\right)\Big]dw_r \\
&= Ef_n\left(M_{0,t} - M_{t,s}\right) - n\int_0^s E_r\Big[\mathbb{I}_{[-\frac{1}{n},0]}\left(M_{0,t} - M_{t,s}\right)\mathbb{I}_{\gamma_{0,t}<r\leq\gamma_{t,s}}\Big]dw_r \\
&= Ef_n\left(M_{0,t} - M_{t,s}\right) - n\int_0^s P_r\left[-\frac{1}{n} \leq M_{0,t} - M_{t,s} \leq 0 \text{ and } \gamma_{0,t} < r \leq \gamma_{t,s}\right]dw_r
\end{aligned}$$

where the second equality follows by the chain rule, the third equality follows by the derivative of f_n, and the fourth equality follows from the product of indicator functions.

We next compute the conditional probability under the integral. This integral is conditioned on the information at time r. We have that $r < s$, but need to consider explicitly the cases where $r < t$ and the case where $r \geq t$.

We start by considering the case $r \leq t$ and hence that $r \leq \gamma_{t,s}$ as $s > t$. In this scenario, the condition probability simplifies to $P_r\left[-\frac{1}{n} \leq M_{0,t} - M_{t,s} \leq 0 \text{ and } \gamma_{0,t} < r\right]$. Since $r \leq t$, and the maximum on interval $[0,t]$ is attained before r, or $\gamma_{0,t} < r$, we have $M_{0,t} = M_{0,r}$ and $M_{r,t} \leq M_{0,r}$ so that the conditional probability can be written as:

$$\begin{aligned}
&P_r\left[-\frac{1}{n} \leq M_{0,t} - M_{t,s} \leq 0 \text{ and } \gamma_{0,t} < r\right] = P_r\left[-\frac{1}{n} \leq M_{0,r} - M_{t,s} \leq 0 \text{ and } M_{r,t} \leq M_{0,r}\right] = \\
&P_r\left[-\frac{1}{n} \leq M_{0,r} - w_t - (M_{t,s} - w_t) \leq 0 \text{ and } M_{r,t} \leq M_{0,r}\right] = \\
&P_r\left[M_{0,r} - w_t + \frac{1}{n} \geq M_{t,s} - w_t \geq M_{0,r} - w_t \text{ and } M_{r,t} \leq M_{0,r}\right]
\end{aligned}$$

We further observe that as $r \leq t < s$ it follows that $M_{t,s} - w_t = \max_{q \in [0,s-t]} (w_{t+q} - w_t)$ is a maximum $\tilde{M}_{0,s-t}$ of a Brownian motion $\tilde{w}_q = w_{t+q} - w_t$. The density of the maximum of a Brownian motion $\tilde{M}_{0,s-t}$ is $\psi_{s-t}(z) = \sqrt{\frac{2}{\pi(s-t)}} e^{-\frac{z^2}{2(s-t)}}$. Therefore, the conditional probability is:

$$E_r \left[\mathbb{I}_{\{M_{0,r} \geq M_{r,t}\}} \int_{M_{0,r} - w_t}^{M_{0,r} - w_t + \frac{1}{n}} \psi_{s-t}(x) dx \right]$$

Next, we consider the case $r > t$, in this case $r > \gamma_{0,t}$ and the conditional probability simplifies to $P_r \left[-\frac{1}{n} \leq M_{0,t} - M_{t,s} \leq 0 \text{ and } r \leq \gamma_{t,s} \right]$. Since $r > t$, and the maximum on interval $[t, s]$ is attained after r, or $r \leq \gamma_{t,s}$, we have that $M_{t,s} = M_{r,s}$ and that $M_{t,r} \leq M_{r,s}$. As a result, the conditional probability can be written as:

$$P_r \left[-\frac{1}{n} \leq M_{0,t} - M_{t,s} \leq 0 \text{ and } r \leq \gamma_{t,s} \right] = P_r \left[-\frac{1}{n} \leq M_{0,t} - M_{r,s} \leq 0 \text{ and } M_{t,r} \leq M_{r,s} \right] =$$
$$P_r \left[M_{0,t} + \frac{1}{n} \geq M_{r,s} \geq M_{0,t} \text{ and } M_{t,r} \leq M_{r,s} \right] = P_r \left[M_{0,t} + \frac{1}{n} \geq M_{r,s} \geq \max(M_{0,t}, M_{t,r}) \right] =$$
$$P_r \left[M_{0,t} + \frac{1}{n} \geq M_{r,s} \geq M_{0,r} \right] = P_r \left[M_{0,t} - w_r + \frac{1}{n} \geq M_{r,s} - w_r \geq M_{0,r} - w_r \right]$$

We further observe that as $t < r < s$ it follows that $M_{r,s} - w_r = \max_{q \in [0,s-r]} (w_{r+q} - w_r)$ is a maximum $\tilde{M}_{0,s-r}$ of a Brownian motion $\tilde{w}_q = w_{r+q} - w_r$. Hence, the probability can be written as:

$$\mathbb{I}_{\{M_{0,t} + \frac{1}{n} \geq M_{0,r}\}} \int_{M_{0,r} - w_r}^{M_{0,t} - w_r + \frac{1}{n}} \psi_{s-r}(x) dx$$

where we observe that no expectations appear here as $r > t$, and hence all information about time t is known at time r.

Having separately computed the conditional probability under the integral, we can now write the Clark representation as:

$$\begin{aligned} f_n(M_{0,t} - M_{t,s}) &= E f_n \left(M_{0,t} - M_{t,s} \right) - n \int_0^s P_r \left[-\frac{1}{n} \leq M_{0,t} - M_{t,s} \leq 0 \text{ and } \gamma_{0,t} < r \leq \gamma_{t,s} \right] dw_r \\ &= E f_n \left(M_{0,t} - M_{t,s} \right) - n \int_0^t E_r \left[\mathbb{I}_{\{M_{0,r} \geq M_{r,t}\}} \int_{M_{0,r} - w_t}^{M_{0,r} - w_t + \frac{1}{n}} \psi_{s-t}(x) dx \right] dw_r \\ &- n \int_t^s \left[\mathbb{I}_{\{M_{0,r} \leq M_{0,t} + \frac{1}{n}\}} \int_{M_{0,r} - w_r}^{M_{0,t} - w_r + \frac{1}{n}} \psi_{s-r}(x) dx \right] dw_r. \end{aligned}$$

When $r > t$ and $M_{0,r} > M_{0,t}$, then the integrand in the second integral is zero for large enough n. Therefore, in the case $r > t$, we only consider the case $M_{0,r} = M_{0,t}$ with the understanding that we consider large enough n:

$$f_n(M_{0,t}-M_{t,s}) = Ef_n\left(M_{0,t}-M_{t,s}\right) - n\int_0^t E_r\left[\mathbb{I}_{\{M_{0,r}\geq M_{r,t}\}}\int_{M_{0,r}-w_t}^{M_{0,r}-w_t+\frac{1}{n}}\psi_{s-t}(x)dx\right]dw_r$$
$$- n\int_t^s\left[\mathbb{I}_{\{M_{0,r}=M_{0,t}\}}\int_{M_{0,r}-w_r}^{M_{0,t}-w_r+\frac{1}{n}}\psi_{s-r}(x)dx\right]dw_r.$$

Taking the limit as $n\to\infty$, we obtain:

$$\mathbb{I}_{\{\gamma_s\leq t\}} = P(\gamma_s\leq t) - \int_0^t E_r\left[\mathbb{I}_{\{M_{0,r}\geq M_{r,t}\}}\psi_{s-t}(M_{0,r}-w_t)\right]dw_r - \int_t^s \mathbb{I}_{\{M_{0,r}=M_{0,t}\}}\psi_{s-r}(M_{0,r}-w_r)dw_r$$
$$= P(\gamma_s\leq t) - \int_0^T\left(\mathbb{I}_{\{r\leq t\}}E_r\left[\mathbb{I}_{\{M_{0,r}\geq M_{r,t}\}}\psi_{s-t}(M_{0,r}-w_t)\right] + \mathbb{I}_{\{t<r<s\}}\mathbb{I}_{\{M_{0,r}=M_{0,t}\}}\psi_{s-r}(M_{0,r}-w_r)\right)dw_r.$$

Given the computation of the conditional expectation, we write the Clark representation of the indicator $\mathbb{I}_{\{\gamma_s\leq t\}}$ as:

$$\mathbb{I}_{\{\gamma_s\leq t\}} = P(\gamma_s\leq t) - \int_0^T\Bigg(\mathbb{I}_{\{r\leq t\}}E_r\left[\mathbb{I}_{\{M_{0,r}\geq M_{r,t}\}}\psi_{s-t}(M_{0,r}-w_t)\right]$$
$$+ \mathbb{I}_{\{t<r<s\}}\mathbb{I}_{\{M_{0,r}=M_{0,t}\}}\psi_{s-r}(M_{0,r}-w_r)\Bigg)dw_r.$$

We next use the Clark representation of the indicator to write the Clark representation of ξ_t. Specifically, we substitute the above expression into (A.19) to obtain:

$$\xi_t = E\xi_t - \int_0^T\Bigg[\int_t^T\int_0^t f'(x)\Bigg(\mathbb{I}_{\{r\leq t-x\}}E_r\left[\mathbb{I}_{\{M_{0,r}\geq M_{r,t-x}\}}\psi_{s-t+x}(M_{0,r}-w_{t-x})\right]$$
$$+ \mathbb{I}_{\{t-x<r<s\}}\mathbb{I}_{\{M_{0,r}=M_{0,t-x}\}}\psi_{s-r}(M_{0,r}-w_r)\Bigg)dxds\Bigg]dw_r,$$

Which finally identifies the diffusion coefficient g_t^t as:

$$g_t^t = \int_t^T\int_0^t f'(x)\mathbb{I}_{\{M_{0,t}=M_{0,t-x}\}}\psi_{s-t}(M_{0,t}-w_t)dxds = \int_t^T\int_0^t f'(x)\mathbb{I}_{\{\gamma_t\leq t-x\}}\psi_{s-t}(M_{0,t}-w_t)dxds$$

This step concludes the full description of the drift and diffusion decomposition in terms of model primitives.

A.7 Quantitative Analysis

This appendix provides more detail on the quantitative analysis, and shows the robustness of the computational results.

A.7.1 Computation

The planner chooses adapted processes for consumption c_t, and hours n_t to maximize welfare (2) given habits (3) subject to the period resource constraints:

$$x_t(w^t) \leq \theta_t(w^t)n_t(w^t), \tag{A.27}$$

for every shock trajectory w^t. We let the Lagrange multiplier corresponding to the period resource constraint (A.27) be denoted $\lambda_t(w^t)$, so the Lagrangian is:

$$E\int_0^\infty \Big(u(x_t(w^t)) - \int_0^t k_t^s(w^t)x_s(w^s)ds - n_t(w^t) + \lambda_t(w^t)\big(\theta_t(w^t)n_t(w^t) - x_t(w^t)\big)\Big)dt. \tag{A.28}$$

We derive optimality conditions after changing the order of integration on the habit in (A.28), $E\int_0^\infty \int_0^t k_t^s(w^t)x_s dsdt = E\int_0^\infty x_t E_t \int_t^\infty k_s^t(w^s)dsdt$. We obtain optimality conditions for habit consumption and labor hours:

$$\lambda_t(w^t) = u'(x_t(w^t)) - E_t\int_t^\infty k_s^t(w^s)ds \qquad \text{and} \qquad \lambda_t(w^t)\theta_t = 1. \tag{A.29}$$

We observe that the optimality condition for labor hours implies that the multiplier only varies with contemporaneous productivity, or $\lambda_t(w^t) = 1/\theta_t = \lambda(\theta_t)$.[41] The first-order condition for consumption is thus one equation in one unknown for $c_t(w^t)$ once we know $E_t\int_t^\infty k_s^t(w^s)ds$.

Laissez-Faire Equilibrium. In the laissez-faire equilibrium, households do not internalize the effect of their consumption onto the future habits of others. Under this specification, the first-order condition for hours is identical to the first-order condition for hours under the planning problem (A.29), while the first-order condition for consumption ignores the effect on habit formation, or:

$$\lambda_t(w^t) = u'(x_t(w^t))\ .$$

Since the multiplier is identical for every trajectory in the two economies, and since the effect of habit consumption on future habits is positive under unfamiliarity, households overconsume.

Equilibrium with Taxes. In an equilibrium with corrective consumption taxes, the optimality condition with respect to hours is again identical to the first-order condition for hours under the

[41] Substituting this condition into the Lagrangian we obtain $E\int_0^\infty \big(u(c_t(w^t)) - \int_0^t k_t^s(w^t)c_s(w^s)ds - \frac{c_t(w^t)}{\theta_t(w^t)}\big)dt$, which shows that the problem can also be written as a planning problem with resource cost shocks.

planning problem and the laissez-fair equilibrium (A.29). Letting the corrective habit consumption tax after shock trajectory w^t be denoted by $\tau_t(w^t)$, the first-order condition for consumption is:

$$\lambda_t(w^t) + \tau_t(w^t)\lambda_t(w^t) = u'(x_t(w^t)) \ .$$

Welfare Calculations. Welfare calculations are based on 10,000 randomly generated sequences of productivities, where each sequence has a length of 1,000 periods that represent 100 years. Welfare levels are obtained by discarding the first 1,000 periods in each sequence and averaging over all 10,000 runs.

A.7.2 Quantitative Results

Table A.1 shows the sensitivity of our quantitative results for different configurations with respect to:

1. The curvature on consumption in the utility function γ;
2. The importance of habits α;
3. The standard deviation of the innovation σ.

A.8 Robustness

This appendix shows the derivations for the policy design problem with concern for robustness.

Our initial problem is to maximize the expectation of the objective:

$$\mathcal{U}(w) = \int_0^T \left[u(c_t) + u(x_t) - \int_0^t k_t^s x_s ds - v(n_t) \right] dt = \int_0^T \left[u(c_t) + u(x_t) - x_t \int_t^T k_s^t ds - v(n_t) \right] dt$$

where the second equality follows by changing the order of integration. Maximization is subject to the resource constraint (1), where expectations are taken with respect to μ, the distribution of the driving Brownian motion w.

We assume the driving process w may be perturbed in a non-detectable way. That is, instead of maximizing $E_\mu \mathcal{U}(w)$, we maximize $E_\mu \mathcal{U}(w^{(h)})$ where:

$$w_t^{(h)} = w_t + \int_0^t h_s ds,$$

and h is some adapted process such that the implied distribution $\mu^{(h)}$ associated with $w^{(h)}$ "is close" to distribution μ associated with w. We measure proximity by the relative entropy $R(\mu^{(h)}||\mu) = E_{\mu^{(h)}} \log \frac{d\mu^{(h)}}{d\mu}$.

Table A.1: Sensitivity of Welfare Loss Under Suboptimal Policy

σ	γ	α	$\kappa = 0$	$\kappa = 1$	
			Δ	Δ_κ	Δ
0.01	1.5	0.2	0.7	0.1	0.5
		0.5	3.4	0.6	2.4
		0.8	7.4	1.1	5.3
0.01	5.0	0.2	0.2	0.0	0.1
		0.5	0.9	0.2	0.7
		0.8	1.8	0.4	1.4
0.04	1.5	0.2	0.7	0.4	0.2
		0.5	3.5	1.6	1.2
		0.8	7.4	3.1	2.6
0.04	5.0	0.2	0.2	0.1	0.1
		0.5	0.9	0.5	0.3
		0.8	1.8	1.0	0.7

Table A.1 shows the robustness of the quantitative results to different configurations for the importance of habits α, the curvature on consumption in preferences γ, and the standard deviation of the innovation σ. The depreciation rate is set to $\delta = 0.03$.

To capture the worst possible case we first minimize $E_\mu \mathcal{U}(w^{(h)})$ subject to $R(\mu^{(h)}||\mu) \leq A$. The Lagrangian of the minimization problem can be written as:

$$E_\mu \mathcal{U}(w^{(h)} + \frac{1}{\kappa}\big(R(\mu^{(h)}||\mu) - A\big),$$

where $\kappa > 0$. Proposition 1.4.2 in Dupuis and Ellis (2011) characterizes the minimum value as:

$$\inf_h \Big[E_\mu \mathcal{U}(w^{(h)}) + \frac{1}{\kappa}\big(R(\mu^{(h)}||\mu) - A\big)\Big] = -\frac{1}{\kappa}\log E_\mu e^{-\kappa \mathcal{U}(w)}.$$

As a result, after minimizing with respect to perturbation h we obtain the problem of maximizing $-\frac{1}{\kappa}\log E_\mu e^{-\kappa U(w)}$ subject to the budget constraint. The Lagrangian is

$$-\frac{1}{\kappa}\log E_\mu e^{-\kappa \mathcal{U}(w)} + \lambda E_\mu \int_0^T (\theta_t n_t - c_t - x_t) dt$$

We next analyze the optimality conditions for this problem. The first-order conditions with respect to consumption and labor supply are given by:

$$\lambda = u'(c_t)\frac{E_t e^{-\kappa U(w)}}{E_\mu e^{-\kappa U(w)}} \qquad \text{and} \qquad \lambda\theta_t = v'(n_t)\frac{E_t e^{-\kappa U(w)}}{E_\mu e^{-\kappa U(w)}}$$

Finally, the first-order condition with respect to habit consumption is given by:

$$\lambda = \frac{E_t e^{-\kappa U(w)}\left(u'(x_t) - \int_t^T k_s^t ds\right)}{E_\mu e^{-\kappa U(w)}}$$

Combining the optimality condition for consumption and habit consumption, optimal policy (5) is given by:

$$\Lambda_t^R = \frac{E_t \int_t^T e^{-\kappa U(w)} k_s^t ds}{E_t e^{-\kappa U(w)}} \tag{12}$$

Robustness considerations modify the habit coefficients k_s^t by multiplying them with $\zeta = e^{-\kappa U(w)}$ and normalizing by $M_t = E_t e^{-\kappa U(w)} = E_t \zeta$. Since the denominator M_t is a martingale, we cannot expect that it is differentiable with respect to time, but we can write $dM_t = E_t\left[D_t\zeta\right]dw_t$.

For the process in the numerator, we write the drift and diffusion decomposition. By the total derivative formula (8),

$$d\left(E_t\zeta\int_t^T k_s^t ds\right) = \left[-k_t^t E_t\zeta + E_t\zeta\int_t^T \partial_t k_s^t ds\right]dt + \left[E_t\int_t^T \left(k_s^t D_t\zeta + \zeta D_t k_s^t\right)ds\right]dw_t.$$

Finally, we apply the Ito formula to obtain the drift and diffusion coefficient for optimal robust policy. After grouping terms, we obtain:

$$\begin{aligned} d\Lambda_t^R = &\left[\frac{E_t\zeta\int_t^T \partial_t k_s^t ds - k_t^t E_t\zeta}{M_t} - \frac{E_t[D_t\zeta]E_t\int_t^T\left(k_s^t D_t\zeta + \zeta D_t k_s^t\right)ds}{M_t^2} + \frac{\Lambda_t^R\left(E_t[D_t\zeta]\right)^2}{M_t^2}\right]dt \\ &+ \left[\frac{E_t\int_t^T\left(k_s^t D_t\zeta + \zeta D_t k_s^t\right)ds}{M_t} - \frac{\Lambda_t^R E_t[D_t\zeta]}{M_t}\right]dw_t. \end{aligned}$$

A.9 Limited Commitment

This appendix provides detailed derivations for the limited commitment economy in Section 6. The planning problem is to maximize welfare (2) over all w-adapted processes for consumption, habit consumption and hours subject to the feasibility constraint (1) and the participation constraints (17).

We assign multiplier λ to the resource constraint, multiplier μ_r to the participation constraints, and write the Lagrangian for the problem as:

$$\begin{aligned}\mathcal{L} =& E\int_0^T \Big(u(c_t) + u(x_t) - \int_0^t k_t^s x_s ds - v(n_t)\Big) dt + \lambda E \int_0^T \Big(\theta_t n_t - c_t - x_t\Big) dt \\ &+ E\int_0^T \mu_r \Big(\int_r^T \Big(u(c_t) + u(x_t) - \int_0^t k_t^s x_s ds - v(n_t)\Big) dt - U_r\Big) dr,\end{aligned}$$

where we remove the conditional expectation in the summand since the multiplier μ_r is adapted, using the law of iterated expectations.

We change the order of integration on three terms. Specifically, we change the order of integration on the habit terms in the objective function:

$$\int_0^T \int_0^t k_t^s x_s ds dt = \int_0^T \int_s^T k_t^s x_s dt ds = \int_0^T x_t \int_t^T k_s^t ds dt$$

and, similarly, on the habit terms in the participation constraints:

$$\int_0^T \int_r^T \int_0^t \mu_r k_t^s x_s ds dt dr = \int_0^T \int_s^T \int_0^t \mu_r k_t^s x_s dr dt ds = \int_0^T x_t \int_t^T k_s^t \int_0^s \mu_r dr ds dt.$$

Finally, we change the order of integration on the other elements of the participation constraint:

$$\int_0^T \int_r^T \mu_r \big(u(c_t) + u(x_t) - v(n_t)\big) dt dr = \int_0^T \big(u(c_t) + u(x_t) - v(n_t)\big) \int_0^t \mu_r dr dt.$$

By substituting the three changes in the order of integration, we obtain the Lagrangian:

$$\begin{aligned}\mathcal{L} =& E\int_0^T \Bigg(\big(u\left(c_t\right) + u\left(x_t\right) - v\left(n_t\right)\big)\Big(1 + \int_0^t \mu_r dr\Big) - x_t E_t \int_t^T k_s^t \Big(1 + \int_0^s \mu_r dr\Big) ds \Bigg) dt \\ &+ \lambda E\Big[\int_0^T \left(\theta_t n_t - c_t - x_t\right) dt\Big]\end{aligned}$$

To obtain the optimality conditions, we vary consumption and labor:

$$\lambda = u'(c_t)\Big(1 + \int_0^t \mu_r dr\Big),$$

while the first-order condition for the disutility from labor is given by:

$$\lambda\theta_t = v'(n_t)\Big(1 + \int_0^t \mu_r dr\Big).$$

The first-order condition with respect to habit consumption is given by:

$$\lambda = u'(x_t)\Big(1 + \int_0^t \mu_r dr\Big) - E_t \int_t^T k_s^t \Big(1 + \int_0^s \mu_r dr\Big) ds.$$

Using the first-order condition with respect to consumption, this is equivalent to:

$$u'(c_t) = u'(x_t) - E_t \int_t^T k_s^t \frac{u'(c_t)}{u'(c_s)} ds.$$

A.10 Private Information

This appendix provides derivations for the dynamic private information economy in Section 7.

A.10.1 Incentive Constraint

We derive the local incentive compatibility constraint. Consider a perturbation $\hat{w}_t^\varepsilon = w_t + \varepsilon \int_0^t z_r dr$, where z_r is any adapted process. The induced trajectory is $\hat{w}^{\varepsilon,t}$. We evaluate individual welfare:

$$E \int_0^T \left(u(c_t(\hat{w}^{\varepsilon,t})) + u(x_t(\hat{w}^{\varepsilon,t})) - \int_0^t k_t^s(w^t) x_s(\hat{w}^{\varepsilon,s}) ds - v\left(\frac{y_t(\hat{w}^{\varepsilon,t})}{\theta_t(w^t)} \right) \right) dt$$

which is maximized when $\varepsilon = 0$. We differentiate an agent's objective with respect to ε to obtain:

$$0 = E \int_0^T \left(u'(c_t) \int_0^t D_r c_t z_r dr + u'(x_t) \int_0^t D_r x_t z_r dr - \int_0^t k_t^s \int_0^s D_r x_s z_r dr ds \right.$$
$$\left. - v'\left(\frac{y_t}{\theta_t}\right) \frac{\int_0^t D_r y_t z_r dr}{\theta_t} \right) dt. \tag{A.30}$$

We next rewrite the incentive constraint using the Malliavin derivative for individual welfare. We omit explicit dependence of the allocations on history w^t when there is limited risk of confusion.

Malliavin Derivative for Individual Welfare. To find the Malliavin derivative for individual welfare, we evaluate the impact of a variation of the stochastic process on period utility. That is, we consider $\big(\mathcal{U}_t(\hat{w}^{\varepsilon,t}) - \mathcal{U}_t(w^t)\big)/\varepsilon$ for small ε, which gives:

$$u'(c_t) \int_0^t D_r c_t z_r dr + u'(x_t) \int_0^t D_r x_t z_r dr - \int_0^t k_t^s \int_0^s D_r x_s z_r dr ds - \int_0^t x_s \int_0^t D_r k_t^s z_r dr ds$$
$$- v'\left(\frac{y_t}{\theta_t}\right) \frac{\int_0^t D_r y_t z_r dr}{\theta_t} + v'\left(\frac{y_t}{\theta_t}\right) \frac{y_t \int_0^t D_r \theta_t z_r dr}{{\theta_t}^2}.$$

To obtain the Malliavin derivative, we reorganize the expressions involving the habit in the total variation to isolate z_r. First, we observe:

$$\int_0^t k_t^s \int_0^s D_r x_s z_r dr ds = \int_0^t \int_r^t k_t^s D_r x_s z_r ds dr = \int_0^t \left(\int_r^t k_t^s D_r x_s ds \right) z_r dr \tag{A.31}$$

by changing the order of integration. Second, we observe:

$$\int_0^t x_s \int_0^t D_r k_t^s z_r dr ds = \int_0^t \int_0^t x_s D_r k_t^s z_r ds dr = \int_0^t \left(\int_0^t x_s D_r k_t^s ds \right) z_r dr. \tag{A.32}$$

We use these two terms, (A.31) and (A.32), to rewrite the Malliavin derivative as:

$$\int_0^t \left(u'(c_t) D_r c_t + u'(x_t) D_r x_t - \int_r^t k_t^s D_r x_s ds - \int_0^t x_s D_r k_t^s ds - v'\left(\frac{y_t}{\theta_t}\right) \frac{D_r y_t}{\theta_t} + v'\left(\frac{y_t}{\theta_t}\right) \frac{y_t D_r \theta_t}{\theta_t^2} \right) z_r dr.$$

This expression gives the Malliavin derivative for period utility as $D_r\mathcal{U}_t(w^t) = u'(c_t)\, D_r c_t + u'(x_t)\, D_r x_t - \int_r^t k_t^s(w^t) D_r x_s ds - \int_0^t x_s D_r k_t^s(w^t) ds - v'(y_t/\theta_t(w^t)) \frac{D_r y_t}{\theta_t(w^t)} + v'(y_t/\theta_t(w^t)) \frac{y_t D_r \theta_t}{\theta_t(w^t)^2}$.

We simplify the first-order incentive constraint (A.30). As with the Malliavin derivative for individual welfare, we reorganize the term concerning habit consumption using equation (A.31):

$$0 = E \int_0^T \int_0^t \left(u'(c_t) D_r c_t + u'(x_t) D_r x_t - \int_r^t k_t^s D_r x_s ds - v'\Big(\frac{y_t}{\theta_t}\Big) \frac{D_r y_t}{\theta_t} \right) z_r dr dt.$$

Second, we use the Malliavin derivative for the period utility to write this incentive constraint as:

$$0 = E \int_0^T \int_0^t \left(D_r \mathcal{U}_t + \int_0^t x_s D_r k_t^s ds - v'\Big(\frac{y_t}{\theta_t}\Big) \frac{y_t D_r \theta_t}{\theta_t^2} \right) z_r dr dt. \tag{A.33}$$

We change the order of integration to write:

$$0 = E \int_0^T \left[\int_r^T \left(D_r \mathcal{U}_t + \int_0^t x_s D_r k_t^s ds - v'\Big(\frac{y_t}{\theta_t}\Big) \frac{y_t D_r \theta_t}{\theta_t^2} \right) dt \right] z_r dr. \tag{A.34}$$

If (A.34) were to hold for any process z, the term in the square brackets has to be identically zero. Instead, as (A.34) has to hold for any adapted process z, we reformulate the incentive constraint using the law of iterated expectations. To do so, we first bring the expectation operator under the integral, and use the law of iterated expectations to write:

$$0 = \int_0^T E\left[E_r \left[\int_r^T \left(D_r \mathcal{U}_t + \int_0^t x_s D_r k_t^s ds - v'\Big(\frac{y_t}{\theta_t}\Big) \frac{y_t D_r \theta_t}{\theta_t^2} \right) dt z_r \right] \right] dr.$$

In the conditional expectation, we condition on the information up until time r, implying that z_r is known and that it can be taken out of the conditional expectation to obtain:

$$0 = E \int_0^T E_r \left[\int_r^T \left(D_r \mathcal{U}_t + \int_0^t x_s D_r k_t^s ds - v'\Big(\frac{y_t}{\theta_t}\Big) \frac{y_t D_r \theta_t}{\theta_t^2} \right) dt \right] z_r dr.$$

Therefore, for this equality to hold for arbitrary adapted process z, it has to be that:

$$0 = E_r \int_r^T \left(D_r \mathcal{U}_t + \int_0^t x_s D_r k_t^s ds - v'\Big(\frac{y_t}{\theta_t}\Big) \frac{y_t D_r \theta_t}{\theta_t^2} \right) dt. \tag{A.35}$$

Planning Problem. Having obtained the first-order incentive compatibility constraint (A.35), the planning problem is to maximize aggregate welfare (2) over all adapted processes for consumption, habit consumption, and labor. Maximization is subject to the incentive constraint (A.35) for any trajectory w^t, and subject to the resource constraint (1).

Using $\mu_t(w^t)$ to denote the multiplier on the first-order incentive constraint, and λ to denote the multiplier on the resource constraint, the Langrangian becomes:

$$\mathcal{L} = E\int_0^T \left[\mathcal{U}_t + \lambda(y_t - c_t - x_t) + \mu_t E_t \int_t^T \left(D_t\mathcal{U}_r + \int_0^r x_s D_t k_r^s ds - v'\Big(\frac{y_r}{\theta_r}\Big)\frac{y_r D_t\theta_r}{\theta_r^2}\right) dr\right] dt.$$

To analyze the problem further, we reorganize the incentive component. First, using the law of iterated expectations, this incentive component is written as:

$$E\int_0^T \int_t^T \left(D_t\mathcal{U}_r + \int_0^r x_s D_t k_r^s ds - v'\Big(\frac{y_r}{\theta_r}\Big)\frac{y_r D_t\theta_r}{\theta_r^2}\right) dr \mu_t dt$$

which is equivalent to the right-hand side of equation (A.34). Thus, the expression is equivalent to (A.33) after changing the order of integration. As a result, we can write the Lagrangian as:

$$\mathcal{L} = E\int_0^T \left[\mathcal{U}_t + \lambda\big(y_t - c_t - x_t\big) + \int_0^t \left(D_r\mathcal{U}_t + \int_0^t x_s D_r k_t^s ds - v'\Big(\frac{y_t}{\theta_t}\Big)\frac{y_t D_r\theta_t}{\theta_t^2}\right) \mu_r dr\right] dt$$

We further apply the Malliavin integration by parts (22) to the period utility function $\mathcal{U}_t$ to write $E\big[\int_0^t D_r\mathcal{U}_t(w^t)\mu_r dr\big] = E\big[\mathcal{U}_t(w^t)\int_0^t \mu_r dw_r\big]$ giving the Lagrangian as:

$$E\int_0^T \left[\mathcal{U}_t\left(1 + \int_0^t \mu_r dw_r\right) + \lambda\big(y_t - c_t - x_t\big) + \int_0^t \left(\int_0^t x_s D_r k_t^s ds - v'\Big(\frac{y_t}{\theta_t}\Big)\frac{y_t D_r\theta_t}{\theta_t^2}\right) \mu_r dr\right] dt.$$

Substituting the habit term using (A.32), we obtain:

$$E\int_0^T \left[\mathcal{U}_t\left(1 + \int_0^t \mu_r dw_r\right) + \lambda\big(y_t - c_t - x_t\big) - v'\Big(\frac{y_t}{\theta_t}\Big)\frac{y_t}{\theta_t^2}\int_0^t D_r\theta_t\mu_r dr + \int_0^t x_s \int_0^t D_r k_t^s \mu_r dr ds\right] dt,$$

Finally, we change the order of integration on the habit terms between t and s:

$$\int_0^T \int_0^t x_s \int_0^t D_r k_t^s \mu_r dr ds dt = \int_0^T \int_s^T x_s \int_0^t D_r k_t^s \mu_r dr dt ds = \int_0^T x_s \int_s^T \int_0^t D_r k_t^s \mu_r dr dt ds$$

Interchanging labels of t and s and substituting into the Lagrangian, where we expand the period utility function, we obtain:

$$\begin{aligned}\mathcal{L} = & E\int_0^T \left(\left(u(c_t) + u(x_t) - \int_0^t k_t^s x_s ds - v\left(\frac{y_t}{\theta_t}\right)\right)\left(1 + \int_0^t \mu_r dw_r\right) + \lambda\big(y_t - c_t - x_t\big)\right) dt \\ & - E\int_0^T v'\Big(\frac{y_t}{\theta_t}\Big)\frac{y_t}{\theta_t^2}\int_0^t D_r\theta_t\mu_r dr dt + E\int_0^T x_t \int_t^T \int_0^s D_r k_s^t \mu_r dr ds dt,\end{aligned}$$

which is equivalent to Lagrangian in the body of the text (24) when we abstract from the terms associated with habit consumption. We use this formulation to obtain the optimality conditions.

Optimality Conditions. We vary consumption, habit consumption, and labor earnings. The first-order condition with respect to consumption is (25), while the first-order condition for the disutility from labor is given by:

$$\lambda = v'\Big(\frac{y_t}{\theta_t}\Big)\frac{1}{\theta_t}\left(1+\int_0^t \mu_r dw_r\right) + v'\Big(\frac{y_t}{\theta_t}\Big)\frac{1}{\theta_t^2}\int_0^t D_r\theta_t\mu_r dr + v''\Big(\frac{y_t}{\theta_t}\Big)\frac{y_t}{\theta_t^3}\int_0^t D_r\theta_t\mu_r dr. \tag{A.36}$$

Finally, we consider the first-order condition with respect to habit consumption to write:

$$\lambda = u'(x_t)\left(1+\int_0^t \mu_r dw_r\right) + E_t\int_t^T\int_0^s D_r k_s^t \mu_r dr ds - E_t\int_t^T k_s^t\left(1+\int_0^s \mu_r dw_r\right)ds. \tag{A.37}$$

A.10.2 Martingale

We next establish that $m(c_t) = \lambda/u'(c_t)$ is a martingale, and derive the characterization of the multipliers on the incentive constraints (27).

First, we rewrite the optimality condition for consumption (25) to see $m(c_t)$ is a martingale:

$$\frac{\lambda}{u'(c_t)} = 1 + \int_0^t \mu_r dw_r.$$

so that $m(c_0) = 1$, and $c_s = E_s m(c_t)$ for all $s \leq t$.

Since $m(c_t)$ is a martingale random variable, its Clark-Ocone representation can be written as:

$$\frac{\lambda}{u'(c_t)} = Em(c_t) + \int_0^t E_r\big[D_r m(c_t)\big]dw_r = 1 + \int_0^t E_r\big[D_r m(c_t)\big]dw_r,$$

where the second equality uses that $Em(c_t) = m(c_0) = 1$. The characterization of the Lagrange multiplier (27) follows by comparing this expression to the first-order condition for consumption (25).

A.10.3 Labor Wedge

To derive the optimal labor wedge formula, we start with the first-order condition for labor (A.36). Multiplying by θ_t, using $y_t = \theta_t n_t$:

$$\theta_t\lambda = v'(n_t)\left(1+\int_0^t \mu_r dw_r\right) + \big(v'(n_t) + v''(n_t)n_t\big)\int_0^t \frac{D_r\theta_t}{\theta_t}\mu_r dr.$$

Dividing by $v'(n_t)\lambda$, using the first-order condition for consumption (25) and the definition for the elasticity of labor supply $\varepsilon_t = 1 + \frac{v''(n_t)n_t}{v'(n_t)}$:

$$\frac{\theta_t}{v'(n_t)} - \frac{1}{u'(c_t)} = \varepsilon_t\int_0^t \frac{D_r\theta_t}{\theta_t}\frac{\mu_r}{\lambda}dr.$$

Further observing that $\frac{\tau_t^l}{1-\tau_t^l} = \frac{\theta_t u'(c_t)}{v'(n_t)} - 1$, this simplifies to:

$$\frac{\tau_t^l}{1-\tau_t^l} = \varepsilon_t \int_0^t \frac{D_r\theta_t}{\theta_t}\frac{\mu_r}{\lambda}u'(c_t)dr.$$

Using the definition of the stochastic discount factor $m(c_t) = \frac{\lambda}{u'(c_t)}$, and the characterization of the multiplier on the incentive constraint (27) we write that:

$$\frac{\tau_t^l}{1-\tau_t^l} = \varepsilon_t \int_0^t \frac{D_r\theta_t}{\theta_t}\frac{E_r[D_r m(c_t)]}{m(c_t)}dr,$$

which is the labor wedge (29).

A.10.4 Habit Wedge

We next analyze the optimality condition for habit consumption good (A.37). Using the first-order condition for consumption (25), this can be rewritten as:

$$u'(c_t) = u'(x_t) - E_t\int_t^T k_s^t m(c_s, c_t)ds + E_t\int_t^T\int_0^s D_r k_s^t \frac{E_r[D_r m(c_s)]}{m(c_t)}drds$$

where we use the first-order condition with respect to consumption (25) and the characterization of the Lagrange multiplier (27).

We next analyze the evolution of the optimal habit consumption wedge, which is expressed compactly as $\Lambda_t = E_t\int_t^T \left(k_s^t m(c_s,c_t) - \int_0^s D_r k_s^t \frac{E_r[D_r m(c_s)]}{m(c_t)}dr\right)ds$. By applying the total derivative formula (7), we obtain:

$$\begin{aligned} d\Lambda_t = &\left[E_t\int_t^T\left(\partial_t(k_s^t m(c_s,c_t)) - \int_0^s D_r\partial_t k_s^t\frac{E_r[D_r m(c_s)]}{m(c_t)}dr\right)ds - k_t^t + \int_0^t D_r k_t^t\frac{E_r[D_r m(c_t)]}{m(c_t)}dr\right]dt \\ &+ E_t\int_t^T D_t\left(k_s^t m(c_s,c_t) - \int_0^s D_r k_s^t\frac{E_r[D_r m(c_s)]}{m(c_t)}dr\right)dsdw_t, \end{aligned}$$

We derived the decomposition of the effects of habit in terms of the drift and volatility and extended the characterization of Λ_t without incentives in (8) to the case with private information constraints.

A.10.5 Savings Wedge

We next derive the term structure for the intertemporal distortion. Using the definition (26), we can write the savings wedge between period from t and period $t+\delta$ as:

$$\tau_t^\delta = 1 - \frac{E_t u'(c_{t+\delta})}{u'(c_t)} = -\frac{E_t u'(c_{t+\delta}) - u'(c_t)}{u'(c_t)}. \tag{26}$$

We recall the first-order condition with respect to consumption (25), and let $m = \frac{\lambda}{u'(c_t)}$, which by the optimality condition for consumption thus follows an Ito process. Applying the Ito formula to the function $F(m) = \frac{\lambda}{m}$,

$$du'(c_t) = -\frac{u'(c_t)^2 \mu_t}{\lambda} dw_t + \frac{u'(c_t)^3 \mu_t^2}{\lambda^2} dt.$$

We can use this expression in order to write:

$$E_t(u'(c_{t+\delta}) - u'(c_t)) = E_t\Big[-\frac{1}{\lambda}\int_t^{t+\delta} u'(c_s)^2 \mu_s dw_s + \int_t^{t+\delta} u'(c_s)^3 \frac{\mu_s^2}{\lambda^2} ds\Big] = E_t \int_t^{t+\delta} u'(c_s)^3 \frac{\mu_s^2}{\lambda^2} ds.$$

Dividing by the marginal utility of consumption at date t, we obtain:

$$-\tau_t^\delta = E_t\left[\frac{u'(c_{t+\delta}) - u'(c_t)}{u'(c_t)}\right] = E_t \int_t^{t+\delta} \frac{u'(c_s)^3}{u'(c_t)} \frac{\mu_s^2}{\lambda^2} ds = E_t \int_t^{t+\delta} m(c_t, c_s) \left(\frac{E_s[D_s m(c_{t+\delta})]}{m(c_s)}\right)^2 ds$$

The instantaneous savings wedge $\hat{\tau}_t$ is a special case $\hat{\tau}_t = \lim_{\delta \to 0} \tau_t^\delta / \delta$ and describes a short-term intertemporal distortion. Formally, dividing by δ and letting $\delta \to 0$ we get the instantaneous tax at time t:

$$\hat{\tau}_t = -\frac{u'(c_t)^2 \mu_t^2}{\lambda^2}. \tag{A.38}$$

To analyze the dynamics of the intertemporal distortion between periods t and $t+\delta$ (where $\delta > 0$ is fixed), we apply the total derivative formula (7):

$$\begin{aligned} dE_t(u'(c_{t+\delta}) - u'(c_t)) &= \frac{1}{\lambda^2} E_t \partial_t \int_t^{t+\delta} u'(c_s)^3 \mu_s^2 ds dt + \frac{1}{\lambda^2} E_t \int_t^{t+\delta} D_t\left[u'(c_s)^3 \mu_s^2\right] ds dw_t \\ &= \frac{1}{\lambda^2}\left(E_t\left[u'(c_{t+\delta})^3 \mu_{t+\delta}^2\right] - u'(c_t)^3 \mu_t^2\right) dt + \frac{1}{\lambda^2} \int_t^{t+\delta} E_t D_t\left[u'(c_s)^3 \mu_s^2\right] ds dw_t \end{aligned}$$

Hence, by the Ito product rule we analyze $d\tau_t^\delta = -\frac{1}{\lambda} d\Big(E_t\left(u'(c_{t+\delta}) - u'(c_t)\right) \times \frac{\lambda}{u'(c_t)}\Big)$ to obtain:

$$\begin{aligned} d\tau_t^\delta = &-\frac{1}{\lambda}\mu_t E_t\left(u'(c_{t+\delta}) - u'(c_t)\right) dw_t - \frac{1}{\lambda^2}\left(\int_t^{t+\delta} \frac{E_t D_t\left[u'(c_s)^3 \mu_s^2\right]}{u'(c_t)} ds\right) dw_t \\ &- \frac{1}{\lambda^2}\left(E_t\left[\frac{u'(c_{t+\delta})^3 \mu_{t+\delta}^2}{u'(c_t)}\right] - u'(c_t)^2 \mu_t^2\right) dt - \frac{1}{\lambda^3}\mu_t\left(\int_t^{t+\delta} E_t D_t\left[u'(c_s)^3 \mu_s^2\right] ds\right) dt \end{aligned}$$

We simplify this expression by using the instantaneous intertemporal distortion (A.38) to write:

$$\begin{aligned} d\tau_t^\delta = &-\frac{1}{\lambda}\mu_t E_t\left(u'(c_{t+\delta}) - u'(c_t)\right) dw_t + \int_t^{t+\delta} E_t\big[D_t m(c_t, c_s)\hat{\tau}_s\big] ds dw_t + \Big(E_t\big[\hat{\tau}_{t+\delta} m(c_t, c_{t+\delta})\big] - \hat{\tau}_t\Big) dt \\ &+ \mu_t\Big(\int_t^{t+\delta} E_t D_t\left[\hat{\tau}_s / m(c_s)\right] ds\Big) dt \end{aligned}$$

where $\mu_t = E_t\big[D_t m(c_{t+\delta})\big]$ by the characterization of the multiplier (27).

A.11 Incentive Constraint

We consider incentive constraints of the form $U(\hat{w}; w) \leq U(w; w)$, where

$$U(\hat{w}; w) = E \int_0^T \left(u(c_t(\hat{w}^t)) - v\left(\frac{y_t(\hat{w}^t)}{\theta_t(w^t)} \right) \right) dt,$$

$\hat{w}_t = w_t + \varepsilon \int_0^t z_s ds$, and z_t is an adapted process. Our aim is to capture possibilities of misreporting by agents in a way that is not detectable. Formally this means that the distribution of the process $\hat{w}$ is absolutely continuous with respect to the Wiener measure. With our specification, $\hat{w}_t = w_t + \varepsilon \int_0^t z_s ds$, we cover sufficiently many distributions of $\hat{w}$ that are absolutely continuous with respect to the Wiener measure.

Let ρ be the probability density, that is, $\rho \geq 0$ and $E\rho = 1$. Consider a new probability measure $Q(A) = E[\rho 1_A]$. Let $\rho_t = E_t \rho$. By the Clark theorem, $\rho = 1 + \int_0^T h_s dw_s$ for some adapted process h_t. Girsanov theorem states that the distribution of the process $w_t - \int_0^t \frac{h_s(w^s)}{\rho_s(w^s)} ds$ under the measure Q coincides with the distribution of w. If the equation $\hat{w}_t = w_t - \int_0^t \frac{h_s(w^s)}{\rho_s(w^s)} ds$ has a unique solution w, then there exists a process $w_t + \int_0^t z_s ds$ with the distribution Q under the original measure. To be precise, the class of such distributions is dense in the space of all measures that are absolutely continuous with respect to the Wiener measure in the total variation norm (Feyel, Ustünel, and Zakai, 2006).

A.12 Malliavin Derivative of General Diffusion

Consider the solution of the stochastic differential equation:

$$da_t = b(a_t)\, dt + \sigma(a_t)\, dw_t,$$

which is a diffusion since the coefficients $b(a_t)$ and $\sigma(a_t)$ are both state-dependent. Equivalently, we write the diffusion process as:

$$a_t = a_0 + \int_0^t b(a_r) dr + \int_0^t \sigma(a_r) dw_r.$$

By Malliavin differentiating both sides of the diffusion process, using $D_s a_r = 0$ for $r < s$:

$$D_s a_t = \int_s^t b'(a_r) D_s a_r dr + \int_s^t \sigma'(a_r) D_s a_r dw_r + \sigma(a_s).$$

We observe that the Malliavin derivative process $(D_s a_t)_{s \leq t}$ satisfies the equation:

$$d(D_s a_t) = b'(a_t)(D_s a_t) dt + \sigma'(a_t)(D_s a_t) dw_t, \tag{A.39}$$

Table A.2: Example of Diffusion Processes

	Drift $b(a_t)$	Diffusion $\sigma(a_t)$	Malliavin semi-elasticity
Brownian Motion	μ	σ	σ
Ornstein-Uhlenbeck	$-\zeta a_t$	σ	$\sigma \exp(-\zeta(t-s))$
Cox-Ingersoll-Ross	$-\zeta a_t$	$\sigma\sqrt{a_t}$	$\sigma\sqrt{a_s}\exp(-\int_s^t(\zeta+\frac{\sigma^2}{8a_r})dr+\frac{1}{2}\int_s^t\frac{\sigma}{\sqrt{a_r}}dw_r)$

Table A.2 calculates the Malliavin semi-elasticity for three common diffusion processes: the Brownian motion, the Ornstein-Uhlenbeck process, and the Cox-Ingersoll-Ross process. The first two columns show the drift and diffusion coefficient, the final column shows the Malliavin semi-elasticity (A.40).

subject to the initial condition that $D_s a_s = \sigma(a_s)$. This equation shows that the tangent process of a diffusion process is an Ito process. Specifically, it is an Ito process with coefficients proportional to the derivatives of the drift and volatility with the initial point equal to the volatility at time s.

Since the Malliavin derivative follows an Ito process, we can analyze the logarithmic transformation of the Malliavin derivative $\log D_s a_t$ by applying Ito's lemma. Doing so, we directly obtain:

$$d\log D_s a_t = \Big(b'(a_t) - \frac{1}{2}\sigma'(a_t)^2\Big)dt + \sigma'(a_t)dw_t.$$

Integrating this equation from time s to time t, using the initial condition that $D_s a_s = \sigma(a_s)$, we obtain:

$$\log D_s a_t = \log\sigma(a_s) + \Big(\int_s^t \Big(b'(a_r) - \frac{1}{2}\sigma'(a_r)^2\Big)dr + \int_s^t \sigma'(a_r)dw_r\Big).$$

Taking exponents, we obtain:

$$D_s a_t = \sigma(a_s)\exp\Big(\int_s^t \Big(b'(a_r) - \frac{1}{2}\sigma'(a_r)^2\Big)dr + \int_s^t \sigma'(a_r)dw_r\Big). \tag{A.40}$$

Table A.2 calculates the Malliavin semi-elasticity (A.40) for three common diffusion processes.